Lecture 6: Pricing Forwards and Futures

This lecture studies the pricing of forward and futures contracts. We first focus on the similarities of the contracts and derive pricing formulas from market equilibrium and the no-arbitrage principle. We then analyze the differences between the contracts.

I. "Arbitrage-Free" Pricing Principle

II. Forward/Futures Pricing

A. Stocks without Dividends
B. Stocks with Discrete Fixed Dividends
C. Stocks with Continuous Dividend Yield
D. Foreign Currencies (FX)
E. Commodities

III. Summary of Pricing Formulas

A. Financial Forwards and Futures
B. Value of the Forward Contract

The Big Picture

While this lecture is all about pricing forwards and futures, we are introducing one of the most important principles in finance; i.e., arbitrage-free pricing.

- "Arbitrage-free" Pricing Procedure
  - Identify and price a portfolio which replicates the payoffs generated by the forward/futures contract
  - If the value of the replicating portfolio differs from the value of the forward, this represents an *arbitrage* opportunity which investors can profit from by implementing a trading strategy which does not require a cash input but has some positive probability of making profits without risking a loss.
  - E.g., if the value of the replicating portfolio is less (greater) than the value of the forward, then one can earn positive profits with zero risk and zero net investment by selling (buying) forward, buying (shorting) the underlying and borrowing (lending) money. "Arbitrage-free" pricing implies that such profits

The Law of One-Price

- Arbitrage-free pricing is based upon the “law of one price”
  - assets with the same payoff have the same price
- We’ll typically be employing the law of one price throughout the semester to determine arbitrage-free prices for various financial derivatives
  - We’ll look for a portfolio of assets that has exactly the same payoffs as the derivative of interest
  - We’ll price the derivative by pricing the replicating portfolio and invoking the law of one price
  - Note that the replicating portfolio is itself a “synthetic” derivative.
- The procedure for identifying and pricing replicating portfolios is most intuitive with forwards/futures, which is why we start with these.

II. Pricing Forward and Futures

Intuition: The forward price should be the price of **"holding the spot"** until maturity; i.e.,
$F(t,T) = S(t)e^{r(T-t)}$.

- If the price is higher than the cost of holding the spot until delivery, no one will buy forward
  - It's cheaper to get the underlying today
- *Understand* this, and you'll understand forwards!

A. An Example: Stocks without Dividends

- Suppose you hold 10,000 shares of stock priced at \$100/share.
- You can either sell today or commit to selling in six months.
- The risk-free interest rate is 4%, and the term structure is flat
  - Annualized continuously compounded; i.*e.,* the continuously compounded risk-free rate between today and any other month in the future is 4%.
- What is the 6-month forward/futures price of the stock?

The No-Arbitrage Argument

The forward/futures price *must* equal $\$102.02$/share.

- What if the price is $\$103.00$/share?

Transaction	Payoff at t	Payoff at T

- What if the price is $\$101.00$/share?

Transaction	Payoff at t	Payoff at T

Therefore, if

$$F(t, T) \neq S(t) \times \mathrm{e}^{r \times (T-t)}$$

then there is an arbitrage opportunity.

B. Stocks with Discrete Fixed Dividends

Most stocks pay dividends. Assume dividends, or costs (exceptional cash leaving the firm), are fixed and are paid at known times.

Example Continued ...

The stock pays a $\$1$ dividend in 3 months. What is the 6-month forward/futures price?

Now, two factors determine the relationship between the spot and the forward/futures price:

- The cost (including financing) of the immediate purchase.
- Benefits or costs of holding the security between now and the delivery date.

Both affect the cost of “holding the spot”

No-Arbitrage Argument

The forward/futures price *must* equal $\$101.01$/share.

- What if the price is $\$102.00$/share?

Transaction	Payoff at t	Payoff at t_D	Payoff at T

- What if the price is $\$100.00$/share?

Therefore, if

$$F(t,T) \neq \left[S(t) - PV\left(D\right)\right] \times \mathrm{e}^{r\times(T-t)}$$

then there is an arbitrage opportunity.

C. Stocks with Continuous Dividend Yield

Assume dividends, or costs, are proportional to the price of the security and are paid continuously.

Example: 6-month forward/futures on the S&P 500

- Suppose:
  - The spot price is $\$900$
  - The C.C. risk-free rate is $\boldsymbol{r} = 4\%$
  - The annualized C.C. dividend yield is $\boldsymbol{\delta} = 3\%$

What is the price of the 6 month future

<u>Equilibrium Argument</u>

The "underlying security" is *not* 1 share of the index. If you buy 1 share of the index, hold it, and reinvest the dividends proportionally in the index, you will have

$$e^{\delta \times (T-t)} = e^{0.03 \times 0.5} = 1.01511$$

shares of the index at maturity.

Instead, the “underlying security” is $e^{-\delta\times(T-t)}$ shares because at maturity $e^{-\delta\times(T-t)}$ shares yield:

$$e^{-\delta\times(T-t)} \times e^{\delta\times(T-t)} = 1 \text{ share of the index}$$

The cost of the underlying security is:

$$\begin{aligned} S(t) \times e^{-\delta\times(T-t)} &= \$900 \times e^{-0.03\times0.5} \\ &= \$886.60 \end{aligned}$$

Therefore, the index forward/futures price is:

$$\begin{aligned} F(t,T) =& e^{-\delta\times(T-t)} \times S(t) \times e^{r\times(T-t)} \\ =& S(t) \times e^{(r-\delta)\times(T-t)} \\ =& \$904.51356 \end{aligned}$$

No-Arbitrage Argument

The forward/futures price *must* equal \$904.51.

- What if the price is \$905.00?

Transaction	Payoff at t	Payoff at T

- What if the price is \$903.00?

Therefore, if

$$F(t,T) \neq S(t) \times \mathrm{e}^{(r-\delta)\times(T-t)}$$

then there is an arbitrage opportunity.

D. Foreign Currencies (FX)

Foreign currency forwards/futures are very similar to index forwards/futures. A unit of foreign currency can be viewed as a stock that pays a continuous dividend yield equal to the foreign interest rate.

Covered Interest Rate Parity:

After adjusting for exchange rate differentials, interest rates must be equal across countries.

Example: Suppose $r_{\text{US}} = 7.41\%$, $r_{\text{Swiss}} = 8.87\%$, and the exchange rate is \$$0.6667$/SF, or SF 1.5/\$.

What is the price of a 4-month SF forward/futures?

The "underlying security" is not 1 Swiss Franc. Instead, it is SF $e^{-r_{\text{Swiss}} \times (T-t)}$.

No-Arbitrage Argument

The forward/futures price *must* equal \$$0.6634$.

- What if the price is \$$0.65$?

Transaction	Payoff at t	Payoff at T

- What if the price is \$$0.68$?

Transaction	Payoff at t	Payoff at T

Therefore, if

$$F(t,T) \neq S(t) \times \mathrm{e}^{(r_{\mathrm{US}} - r_{\mathrm{Swiss}}) \times (T-t)}$$

then there is an arbitrage opportunity.

E. Commodities *Cost-of-carry* is the physical storage cost of a commodity – the cost of spoilage, the cost of insurance, etc.

The cost-of-carry increases the forward/futures price, relative to the spot price.

- If we know the cost-of-carry until maturity U, we treat it as a negative dividend:

$$F(t,T) = \left[S(t) + PV(U)\right] \times \mathrm{e}^{r\times(T-t)}$$

- Otherwise, we just assume that the cost-of-carry is proportional to the value of the commodity and is paid continuously at a rate u. That is, we treat it as a negative dividend yield:

$$F(t,T) = S(t) \times \mathrm{e}^{(r+u)\times(T-t)}$$

III. Summary of Pricing Formulas

A. Financial Forwards and Futures

Market prices are given by:

$$F(t,T) = V(t) \times \mathrm{e}^{r(t,T)\times(T-t)}$$

where

$$\begin{aligned} F(t,T) &= \text{Forward/futures price.} \\ r(t,T) &= \text{Continuously compounded risk-free interest rate between times } t \text{ and } T. \\ V(t) &= \text{Value of the ``right'' underlying} \\ &= \text{Amount of money required at time } t \text{ for a strategy that generates one unit of the underlying security at time } T. \end{aligned}$$

- The trick is always to find $V(t)$

The “right underlying” or strategies $V(t)$ are:

- Stock without dividends

  Buy one share at time t and hold until time T:

  $$V(t) = S(t)$$

- Stock with known dividends

  Buy one share at time t, borrow the present value of the dividends, pay off the debt with the dividends, and hold until time T:

  $$V(t) = S(t) - \text{PV(D)}$$

- Stock with known dividend yield δ

  Buy $e^{-\delta\times(T-t)}$ shares at time t, reinvest the dividends in stock, and hold until time T:

  $$V(t) = S(t) \times e^{-\delta\times(T-t)}$$

- Foreign currency (FX)

  Buy $\mathrm{e}^{-r_{\text{foreign}} \times (T-t)}$ units of the currency at time t, earn foreign interest, and hold until time T:

$$V(t) = S(t) \times \mathrm{e}^{-r_{\text{foreign}} \times (T-t)}$$

Value of Forward Contract

The "stock price" is the value of the stock; the "forward price" is **not** the value of forward contract.

What is the value of a forward contract?

- Assume no costs or benefits to holding the spot.
- To find the value of a forward contract *after initiation* ($f(t)$), compare the following:

(i) Long one forward contract worth $f(t)$ at time t.

(ii) Long one unit of the underlying security worth $S(t)$ at time t and $K \times \mathrm{e}^{-r \times (T-t)}$ of borrowing.

Both positions pay off $S(T) - K$ at time T, and must have the same value at time t. Therefore, the value of a forward contract at time t is:

$$f(t) = S(t) - K \times \mathrm{e}^{-r \times (T-t)}$$

- The value of the replicating portfolio.