A Simple Model of a Financial Market

Suppose the current price of a riskless bond is $B_t = \$1$, whereas the current price of a (non-dividend paying) stock $S_t = \$1$. In the future ($T - t$ periods from now), the riskless bond will worth $B_T = \$1$ (i.e., the rate of interest is 0%), whereas the stock will be worth $S_{T,g} = \$2$ if the state of the economy is good (state g), and $S_{T,b} = \$.5$ if the state of the economy is bad (state b).

A. What is the arbitrage-free price for a European call option with an exercise price of \$1 which expires at date T?

SOLUTION: We will price the call option via the replicating portfolio approach. The replicating portfolio for a call option involves buying Δ shares of the stock and borrowing β units of the bond. Thus, the value of the replicating portfolio today (at date t) is:

$$V_{RP,t} = \Delta S_t + \beta B_t.$$

At date T, the value of the call option in the good state is $c_{T,g} = \max(0, S_{T,g} - K) = 2 - 1 = 1$, and in the bad state it is $c_{T,b} = \max(0, S_{T,b} - K) = \max(0, .5 - 1) = 0$. The parameters Δ and β are chosen such that the date T payoffs on the replicating portfolio perfectly mimic the date T payoffs on the call option; thus, we have two equations in two unknowns:

$$\begin{array}{l} V_{RP,T,g} = \Delta S_{T,g} + \beta B_T = \Delta 2 + \beta = 1, \text{ and} \\ V_{RP,T,b} = \Delta S_{T,b} + \beta B_T = \Delta .5 + \beta = 0. \end{array}.$$

Solving these equations for Δ and β, we obtain $\Delta = 2/3$ and $\beta = -1/3$. Since both the call and its replicating portfolio produce the same cash flows, it follows that the value of the replicating portfolio *must* be the arbitrage-free price for the call option:

$$c_t = V_{RP,t} = \Delta S_t + \beta B_t = (2/3)1 - (1/3)1 = \$.33.$$

B. What is the arbitrage-free price for a European put option with an exercise price of \$1 which expires at date T?

SOLUTION: We will price the put option via the replicating portfolio approach. The replicating portfolio for a call option involves shorting Δ shares of the stock and lending β units of the bond. Thus, the value of the replicating portfolio today (at date t) is:

$$V_{RP,t} = \Delta S_t + \beta B_t.$$

At date T, the value of the put option in the good state is

$p_{T,g} = \max(0, K - S_{T,g}) = \max(0, 1-2) = 0$, and in the bad state it is $p_{T,b} = \max(0, K - S_{T,b}) = 1 - .5 = .5$. The parameters Δ and β are chosen such that the date T payoffs on the replicating portfolio perfectly mimic the date T payoffs on the put option; thus, we have two equations in two unknowns:

$$\begin{aligned} V_{RP,T,g} &= \Delta S_{T,g} + \beta B_T = \Delta 2 + \beta = 0, \text{ and} \\ V_{RP,T,b} &= \Delta S_{T,b} + \beta B_T = \Delta .5 + \beta = .5. \end{aligned} .$$

Solving these equations for Δ and β, we obtain $\Delta = -1/3$ and $\beta = 2/3$. Since both the put and its replicating portfolio produce the same cash flows, it follows that the value of the replicating portfolio *must* be the arbitrage-free price for the put option:

$$p_t = V_{RP,t} = \Delta S_t + \beta B_t = -(1/3)1 + (2/3)1 = \$.33.$$

C. Show that the put-call parity equation yields the same price for the put option as the replicating portfolio approach.

SOLUTION: Since we know that 1) the price of the call is $c_t = \$.33$, 2) the price of the stock is $S_t = \$1$, and 3) the present value of the exercise price $PV(K) = Ke^{-r(T-t)} = \$1e^{-0} = \$1$, this implies that the arbitrage-free price for the put option is given by the put-call parity equation; i.e.,

$$p_t = c_t + Ke^{-r(T-t)} - S_t = \$.33 + \$1 - \$1 = \$.33.$$