Binomial Option Pricing Model (single-period)

by James R. Garven*
February 24, 2019

1 Introduction

In this teaching note, we introduce a single-period binomial model for pricing call and put options. We assume that options are European; i.e., exercise may only occur on the expiration date. Furthermore, the underlying asset does not pay off at any time other than the expiration date; i.e., no cash flows occur prior to the expiration of the option contract.

We introduce three approaches to pricing call and put options: 1) the Delta Hedging approach, 2) the Replicating Portfolio approach, and 3) the Risk Neutral Valuation approach. Although all three approaches produce the same set of prices, they are worth studying since they shed a number of important insights into the economics of options.

2 Delta Hedging Approach in a Single Period

Under the delta hedging approach, we create a riskless bond by forming a perfectly hedged portfolio consisting of either a call option or a put option and the underlying stock. The price of the stock at the inception of the option contract is S, and over a discrete time interval (δt) going forward, the stock will assume one of the following two values: $S_u = uS$ or $S_d = dS$, where $u > 1$ and $d < 1$. The probability that the stock price will move "up" is p, and the probability that the stock price will move "down" is $1 - p$. Throughout this teaching note, we will assume that $S = \$100$, $u = 1.05$, $d = .95$, $p = .60$, $\delta t = 1/12$ (one month), the exercise price (K) is \$100, and the interest rate (r) is 5%. Figure 1 shows the binomial "tree" for the current (known) stock price and also the future (state-contingent) stock prices:

*James R. Garven is the Frank S. Groner Memorial Chair in Finance and Professor of Finance & Insurance at Baylor University (Address: Foster 320.39, One Bear Place #98004, Waco, TX 76798, telephone: 254-307-1317, e-mail: James_Garven@baylor.edu).

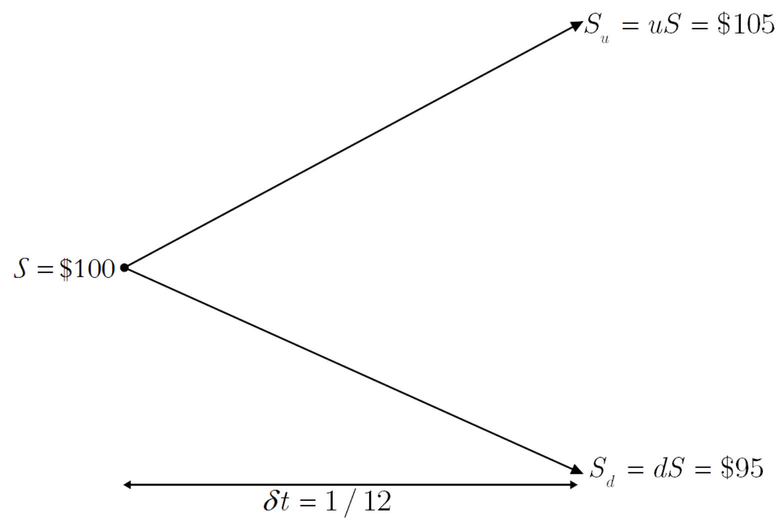

Figure 1: Single-Period Binomial Tree for the Current and Future Stock Prices

Figure 2 shows the binomial "tree" for the current (unknown) call option price and also the future (state-contingent) call option prices:

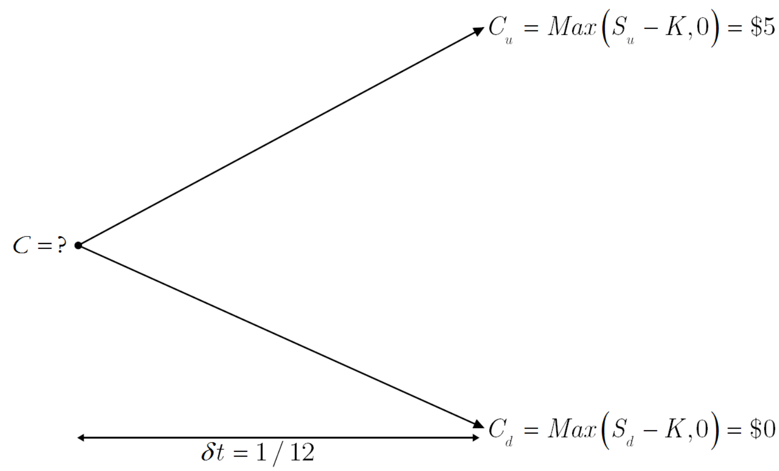

Figure 2: Single-Period Binomial Tree for the Current and Future Call Option Prices

Next, we form a "hedge" portfolio consisting of a long position in one call option and a short position in Δ shares of stock. This portfolio is called a hedge portfolio because movements in the value of the short stock position effectively hedge, or offset the effect of movements in the value of the long call option position. The current value of this portfolio is

$$V_H = C - \Delta S = C - \Delta 100. \tag{1}$$

At node u, the value of the hedge portfolio is equal to $V_H^u = C_u - \Delta S_u = 5 - \Delta 105$, and at node d, the value of the hedge portfolio is equal to $V_H^d = C_d - \Delta S_d = 0 - \Delta 95$. Suppose we select Δ such that the hedge portfolio is riskless; i.e., $V_H^u = V_H^d$. Solving for Δ, we obtain:

$$V_H^u = V_H^d \Rightarrow 5 - \Delta 105 = -\Delta 95 \Rightarrow \Delta = .5. \tag{2}$$

Substituting $\Delta = .5$ into our expressions for V_H^u and V_H^d, we obtain $V_H^u = V_H^d = -\$47.50$. Thus, the value of a riskless hedge portfolio comprised of one call option and a short position in .5 shares of stock is equivalent in value to a *short* position in a riskless bond. In order to prevent arbitrage, the current value of this short bond position, $V_H = C - 50 = -\$e^{-.05/12}47.50 = -\$47.30$, which implies that $C = \$2.70$.[1]

Since we now have the arbitrage-free price for the call option, we can use the put-call parity equation to find the arbitrage-free price of an otherwise identical put option. The put-call parity equation is given by equation (3):

$$C + Ke^{-r\delta t} = P + S. \tag{3}$$

Thus,

$$P = C + Ke^{-r\delta t} - S = \$2.70 + \$100e^{-.05/12} - \$100 = \$2.28. \tag{4}$$

We can also determine the arbitrage-free price for the put option via the delta hedging approach. Since the prices of a put option and its underlying stock are inversely related, we form a hedge portfolio consisting of a long position in one put option and a long position in

[1] Thus, the long call/short stock trading strategy shown here synthetically replicates a *borrowing* transaction; i.e., at the beginning of the period, the investor effectively takes out a loan with a principal value equal to the value of the hedge portfolio and pays back the principal with interest later on. Alternatively, one could also model this problem as a *lending* transaction; this would involve a short call/long stock trading strategy. Showing this is left as an exercise for the reader.

Δ shares of stock. The current value of this portfolio is

$$V_H = P + \Delta S = P + \Delta 100. \tag{5}$$

At node u, the value of the hedge portfolio is equal to $V_H^u = P_u + \Delta S_u = Max(K - 105, 0) + \Delta 105 = 0 + \Delta 105$, and at node d, the value of the hedge portfolio is equal to $V_H^d = P_d + \Delta S_d = Max(K - 95, 0) + \Delta 95 = 5 + \Delta 95$. Suppose we select Δ such that the hedge portfolio is riskless; i.e., $V_H^u = V_H^d$. Solving for Δ, we obtain:

$$V_H^u = V_H^d \Rightarrow \Delta 105 = 5 + \Delta 95 \Rightarrow \Delta = .5. \tag{6}$$

Substituting $\Delta = .5$ into our expressions for V_H^u and V_H^d, we obtain $V_H^u = V_H^d = \$52.50$. Thus, the value of a riskless hedge portfolio comprised of one put option and a long position in .5 shares of stock is equivalent in value to a *long* position in a riskless bond.[2] In order to prevent arbitrage, the current value of this long bond position, $V_H = P + 50 = \$e^{-.05/12}52.50 = \$52.28$, which implies that $P = \$2.28$.

3 Replicating Portfolio Approach in a Single Period

Next, we consider the replicating portfolio approach. The basic setup involves replicating the call option by buying the underlying stock on margin, and replicating the put option by lending money and shorting the underlying stock. By replicating the call and put payoffs at nodes u and d, the current market value of the replicating portfolio must equal the current market value of the option; otherwise investors can earn positive profits with zero risk and zero net investment by buying the less expensive investment and shorting the more expensive one. As in the case of the delta hedging approach, we invoke the no-arbitrage condition to

[2] The same logic applies here as in footnote (1); specifically, one could also model this problem as a *borrowing* transaction which would involve taking short positions in both the put and its underlying stock. As before, showing this is left as an exercise for the reader.

establish that the price of the option must equal the value of its replicating portfolio.

We begin by computing the value of the replicating portfolio for the call option. Suppose we form a portfolio consisting of Δ shares of stock and $\$B$ in riskless bonds. The initial cost of forming such a portfolio is $\$(\Delta S + B)$. One timestep from now,

$$C_u = \Delta u S + e^{r\delta t} B, \text{and} \tag{7}$$

$$C_d = \Delta d S + e^{r\delta t} B. \tag{8}$$

Thus, we have two equations in two unknowns. Solving equations (7) and (8) for Δ and B, we obtain:[3]

$$\Delta = \frac{C_u - C_d}{S(u-d)} \geq 0, \text{and} \tag{9}$$

$$B = \frac{uC_d - dC_u}{e^{r\delta t}(u-d)} \leq 0. \tag{10}$$

Next, let's reconsider these equations in light of our numerical example. From equations (9) and (10), $\Delta = \dfrac{C_u - C_d}{S(u-d)} = 5/10 = .5$ and $B = \dfrac{uC_d - dC_u}{e^{r\delta t}(u-d)} = \dfrac{1.05(0) - .95(5)}{e^{.05/12}(.1)} = -47.30$. Thus, we can replicate the call option by purchasing half of a share for \$50 and borrowing \$47.30. Since the value of the replicating portfolio is $\$(\Delta S + B) = \$50 - 47.30 = \$2.70$, this must also be the value of the call option.

Next, we compute the value of the replicating portfolio for the put option. Suppose we form a portfolio consisting of Δ shares of stock and $\$B$ in riskless bonds. The initial cost of forming such a portfolio is $\$(\Delta S + B)$. One timestep from now,

$$P_u = \Delta u S + e^{r\delta t} B, \text{and} \tag{11}$$

$$P_d = \Delta d S + e^{r\delta t} B. \tag{12}$$

Solving equations (11) and (12) for Δ and B, we obtain:

$$\Delta = \frac{P_u - P_d}{S(u-d)} \leq 0, \text{and} \tag{13}$$

[3]See the appendix which details the algebra used to obtain equations (9), (10), (13) and (14) and provides the intuition behind these equations.

$$B = \frac{uP_d - dP_u}{e^{r\delta t}(u-d)} \geq 0. \tag{14}$$

Next, let's reconsider these equations in light of our numerical example. From equations (13) and (14), $\Delta = \dfrac{P_u - P_d}{S(u-d)} = -5/10 = -.5$ and $B = \dfrac{uP_d - dP_u}{e^{r\delta t}(u-d)} = \dfrac{1.05(5) - .95(0)}{e^{.05/12}(.1)} = 52.28$. Thus, we can replicate the put option by shorting half of a share for \$50 and lending \$52.28. Since the value of the replicating portfolio is $\$(\Delta S + B) = -\$50 + \$52.28 = \$2.28$, this must also be the value of the put option.

4 Risk Neutral Valuation Approach in a Single Period

The final approach that we consider is called the risk neutral valuation approach. In the previous two sections of this teaching note, we obtained arbitrage-free prices for call and put options by either creating a synthetic riskless bond (via the delta hedging approach) or by creating synthetic call and put options (via the replicating portfolio approach). Since the prices of calls, puts, and the underlying stock referenced by these options do not allow for profitable riskless arbitrage, then it does not matter *what* we assume concerning investor risk preferences. Specifically, one can price options assuming *any type* of attitude toward risk and obtain the same answer because discounting under both the delta hedging and replicating portfolio approaches must always occur at the riskless rate of interest.

We start by showing the relationship which exists between the expected return on the underlying stock (μ) and the probability of an up move ($p = 0.60$). Note that

$$E(S_{\delta t}) = puS + (1-p)dS = e^{\mu \delta t} S, \tag{15}$$

where $E(S_{\delta t})$ represents the expected stock price one timestep from today. Solving equation (15) for p, we find that

$$p = \left(e^{\mu \delta t} - d\right) / \left(u - d\right). \tag{16}$$

Since $p = 0.60$, this implies that the annualized expected return on the underlying stock is $\mu = \dfrac{\ln(pu + (1-p)d)}{\delta t} = \dfrac{\ln(.6(1.05) + (.4).95)}{.0833} = 11.94\%$. In a risk averse economy, investors apparently demand an (annualized) expected rate of return on the underlying stock which is 6.94 percentage points higher than the riskless rate of interest.

However, let's suppose that investors are *risk neutral*. This implies that the risk premium on the underlying asset must be 0%, not 6.94% as calculated above. Keeping this in mind, we substitute the riskless rate of interest ($r = 5\%$) in place of the expected rate of return in a risk averse economy ($\mu = 11.94\%$) in the numerator of equation (16) to obtain the *risk neutral* probability (q) of an up move:

$$q = (e^{r\delta t} - d)/(u - d) = (e^{.05/12} - .95)/(.10) = .5418. \tag{17}$$

Since we know the risk neutral probability q, we can calculate the *risk neutral expected value* of the call and put option payoffs 1 timestep (month) from now by simply weighting these payoffs by their corresponding risk neutral probabilities:

$$\hat{E}(C_{\delta t}) = qC_u + (1-q)C_d, \text{and} \tag{18}$$

$$\hat{E}(P_{\delta t}) = qP_u + (1-q)P_d, \text{where} \tag{19}$$

where $\hat{E}(\cdot)$ corresponds to the *risk neutral expected value operator*. Essentially, $\hat{E}(C_{\delta t})$ and $\hat{E}(P_{\delta t})$ represent the certainty-equivalent values for the call and put option payoffs at the expiration date. As such, we can determine the prices of these options by simply discounting these certainty-equivalent values for one month at the riskless rate of interest. Thus, the (arbitrage-free) prices for (single-period) European call and put options are given by the following equations:

$$C = e^{-r\delta t}\hat{E}(C_{\delta t}) = e^{-r\delta t}[qC_u + (1-q)C_d] = e^{-.05/12}[.5418(5)] = \$2.70, \text{and} \tag{20}$$

$$P = e^{-r\delta t}\hat{E}(P_{\delta t}) = e^{-r\delta t}[qP_u + (1-q)P_d] = e^{-.05/12}[.4583(5)] = \$2.28. \tag{21}$$

We conclude this teaching note by showing that risk neutral valuation equations (20) and

(21) are implied by both the delta hedging and replicating portfolio approaches.

5 Risk Neutral Valuation and the Delta Hedging Approach

Let's revisit the delta hedging problem that we solved in section 2 for pricing a call option. There, we formed a hedge portfolio consisting of a long position in one call option and a short position in Δ shares of stock. At the beginning of the binomial tree, the value of the hedge portfolio (as indicated by equation (1)) is $V_H = C - \Delta S$. Since equation (9) indicates that $\Delta = \frac{C_u - C_d}{S(u-d)}$, it follows that

$$V_H = C - \frac{C_u - C_d}{S(u-d)} S = C - \frac{C_u - C_d}{(u-d)}. \tag{22}$$

After one timestep, the value of the hedge portfolio will be the same irrespective of whether the stock moves up or down; i.e., $V_H^u = V_H^d \Rightarrow C_u - \frac{C_u - C_d}{(u-d)} u = C_d - \frac{C_u - C_d}{(u-d)} d$. Thus, the arbitrage-free value of the hedge portfolio, V_H, corresponds to the present value of either V_H^u or V_H^d (let's go with V_H^u); i.e., $V_H = C - \frac{C_u - C_d}{(u-d)} = e^{-r\delta t}\left[C_u - \frac{C_u - C_d}{(u-d)} u\right] \Rightarrow C = \frac{C_u - C_d}{(u-d)} + e^{-r\delta t}\left[C_u - \frac{C_u - C_d}{(u-d)} u\right]$. Solving for the arbitrage-free price of the call option, we find that

$$\begin{aligned} C &= \frac{C_u - C_d + [(u-d)C_u - uC_u + uC_d]e^{-r\delta t}}{u-d} \\ &= \frac{C_u - C_d - dC_u e^{-r\delta t} + uC_d e^{-r\delta t}}{u-d} \\ &= e^{-r\delta t}\left[\frac{e^{r\delta t} - d}{u-d} C_u + \frac{u - e^{r\delta t}}{u-d} C_d\right] \\ &= e^{-r\delta t}\left[qC_u + (1-q)C_d\right]. \end{aligned} \tag{23}$$

Obviously, equation (23) is identical to equation (20). Thus, the delta hedging approach implies that a risk neutral valuation relationship exists between a call option and its underlying stock. By symmetry, the analysis shown here also validates that a risk neutral valuation relationship exists between a put option and its underlying stock (cf. equation (21). Consequently, the delta hedging approach constitutes a sufficient condition for risk neutral valuation.

6 Risk Neutral Valuation and the Replicating Portfolio Approach

The proof of sufficiency for risk neutral valuation is much simpler under the replicating portfolio approach. As shown in the third section of this teaching note, the value of a replicating portfolio $V_{RP} = \Delta S + B$, where $\Delta = \dfrac{C_u - C_d}{S(u-d)}$ and $B = \dfrac{uC_d - dC_u}{e^{r\delta t}(u-d)}$ (cf. equations (9) and (10). Thus,

$$\begin{aligned} C &= \frac{C_u - C_d}{S(u-d)} S + \frac{uC_d - dC_u}{e^{r\delta t}(u-d)} \\ &= \frac{e^{r\delta t}(C_u - C_d) + uC_d - dC_u}{e^{r\delta t}(u-d)} \\ &= e^{-r\delta t} \frac{C_u(e^{r\delta t} - d) + C_d(u - e^{r\delta t})}{(u-d)}. \end{aligned} \tag{24}$$

Since $q = \dfrac{e^{r\delta t} - d}{u-d}$and $1 - q = \dfrac{u - e^{r\delta t}}{u-d}$, substituting q and (1-q) into the right-hand side of equation (24) yields:

$$C = e^{-r\delta t} \left[qC_u + (1-q)C_d \right]. \tag{25}$$

Thus, the replicating portfolio approach implies that a risk neutral valuation relationship exists between a call option and its underlying stock. By symmetry, the analysis shown here also validates that a risk neutral valuation relationship exists between a put option and its underlying stock (cf. equation (21). Consequently, the replicating portfolio approach constitutes a sufficient condition for risk neutral valuation.

Appendix

Here, we detail the algebra used to obtain the expressions for the number of shares and riskless bonds that form the replicating portfolio for the call option (as shown in equations (9) and (10)) and for the put option (as shown in equations (13) and (14)).

Since the algebra is identical for both calls and puts, let f_u and f_d represent the node u and d payoffs for call and put options:

$$f_u = \Delta u S + e^{r\delta t} B, \text{ and} \tag{A.1}$$

$$f_d = \Delta d S + e^{r\delta t} B. \tag{A.2}$$

Subtracting equation (A.2) from (A.1), we obtain:

$$f_u - f_d = \Delta(u-d)S \Rightarrow \Delta = \frac{f_u - f_d}{S(u-d)}. \tag{A.3}$$

Next, we replace Δ in equation (A.1) with $\frac{f_u - f_d}{S(u-d)}$ and solve for B:

$$f_u = \frac{f_u - f_d}{S(u-d)} uS + e^{r\delta t} B \Rightarrow B = \frac{u f_d - d f_u}{e^{r\delta t}(u-d)}. \tag{A.4}$$

In equation (A.3), if we are modeling a call (put) option, then this ratio is unambiguously non-negative (non-positive). Note that the denominator in equation (A.3) is always positive by construction, whereas the numerator can only be greater than or equal (less than or equal) to zero. Furthermore, $f_u - f_d$ can only be zero if options are out-of-the-money at both the u and d nodes. In equation (A.4), the denominator is always positive by construction, whereas the numerator is non-positive (non-negative). Thus we can conclude that long positions in call options represent margined investments in the underlying asset, whereas long positions in put options represent short positions in the underlying asset coupled with a long position in a riskless bond.