Black-Scholes-Merton Call and Put Equations and Comparative Statics

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1 Introduction

This teaching note provides a summary of the Black-Scholes-Merton Option pricing formulas and comparative statics in Section 2, and is followed by numerical simulations in the Section 3.

2 Black-Scholes-Merton Option Pricing Summary

- The Black-Scholes-Merton formula for the value of a European call option is: $c = SN(d_1) - Ke^{-r\tau}N(d_2)$, where $d_1 = \dfrac{\ln(S/K) + (r + .5\sigma^2)\,\tau}{\sigma\sqrt{\tau}}$, $d_2 = d_1 - \sigma\sqrt{\tau}$, and $\tau = T - t$; i.e., τ represents the time to expiration for the option.

- The Black-Scholes-Merton formula for the value of a European put option is $p = Ke^{-r\tau}N(-d_2) - SN(-d_1)$.

- Call and put option prices are functions of 1) the current asset price S, 2) the exercise price K, 3) the rate of interest r, 4) the time to expiration τ, and 5) volatility σ.[1]

- The comparative statics of call and put option prices with respect to S, K, r, t, and σ are as follows:

  - **Asset Price ("delta"):** $\partial c/\partial S = N(d_1) > 0$ and $\partial p/\partial S = -N(-d_1) < 0$.
  - **Exercise Price:** $\partial c/\partial K = -e^{-r\tau}N(d_2) < 0$ and $\partial p/\partial K = e^{-r\tau}N(-d_2) > 0$.
  - **Interest Rate ("rho")**: $\partial c/\partial r = \tau Ke^{-r\tau}N(d_2) > 0$ and $\partial p/\partial r = -\tau Ke^{-r\tau}N(-d_2) < 0$.
  - **Time to Expiration ("theta"):** $\partial c/\partial t = -Sn(d_1)\dfrac{.5\sigma}{\sqrt{\tau}} - rKe^{-r\tau}N(d_2) < 0$ and $\partial p/\partial t = -Sn(d_1)\dfrac{.5\sigma}{\sqrt{\tau}} + rKe^{-r\tau}N(-d_2) <> 0$.

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[1] Option prices are also functionally related to the dividend yield, but in the numerical simulations which follow, we will assume that the underlying asset does not pay dividends.

– **Volatility ("vega"):** $\partial c/\partial\sigma = Sn(d_1)\sqrt{\tau}$ and $\partial p/\partial\sigma = Sn(-d_1)\sqrt{\tau}$. Thus, $\partial c/\partial\sigma = \partial p/\partial\sigma > 0$.

– **Gamma:** $\partial^2 c/\partial S^2 = \partial^2 p/\partial S^2 = n(d_1)/\, S\sigma\sqrt{\tau} > 0$.

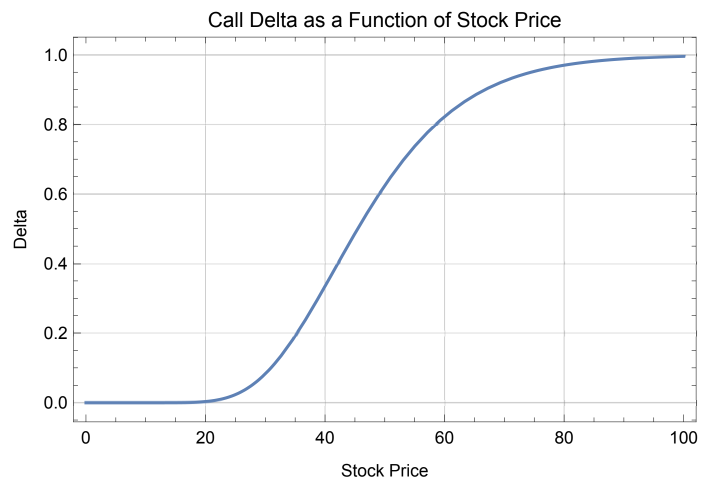
The rest of this teaching note provides numerical simulations of call and put option comparative statics using *Mathematica*. Source code for this teaching note is available at http://fin4366.garven.com/Option_Greeks.nb. In order to use this source code, one must have access to *Mathematica* software. Since Baylor owns a site license for this software, faculty, staff and students can obtain *Mathematica* software by following downloading instructions located at https://www.baylor.edu/helpdesk/index.php?id=980433.

3 Numerical Simulations

The base case for this numerical simulation involves the following set of parameters: 1) the current (date t) stock price is S = \$50, 2) the exercise price is K = \$50, 3) the (continuously compounded) annualized rate of interest is r = 5%, 4) the time to expiration $\tau = T\text{-}t$ = 1 year, and 5) annualized volatility is σ = 30%. The Black-Scholes-Merton price of a call option using these parameters is c = \$7.12, and the corresponding put price is p = \$4.68.

3.1 Relationship between Option Deltas and the price of the underlying

First, we test the relationship between the call option delta and the price of the underlying:



As one would intuitively expect, as the price of the underlying increases, then the hedge ratio converges toward 1. This makes sense because if the option is in-the-money, then the ratio of changes in the option value with respect to changes in the value of the underlying become highly positively correlated. On the other hand, as the price of the underlying diminishes, then the value of the option becomes much less correlated with the value of the underlying; hence the (near zero) hedge ratio in such cases.[2]

Next, we test the relationship between the put hedge ratio and the price of the underlying:

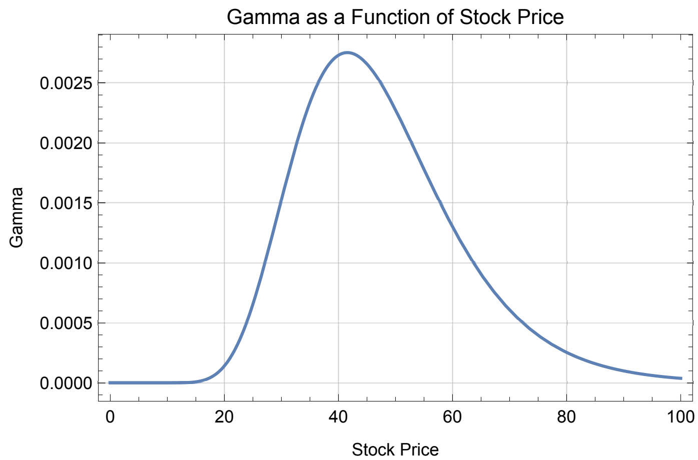
Not surprisingly, the graph for the put looks very much like the graph for the call, except the y axis values are negative. Of course, we expect this since delta for the put option $(-N(-d_1))$ is equal to the call option's delta $(N(d_1))$ minus 1! Here, the hedge ratio for the put option converges toward 0 for deeply out-of-the-money puts (i.e., when the price of the underlying is high), and toward -1 for deeply in-the-money puts.[3]

3.2 Relationship between Option Gammas and the price of the underlying

The call option's gamma is identical to the put option gamma; specifically, $\dfrac{\partial^2 c}{\partial S^2} = \dfrac{\partial^2 p}{\partial S^2} = n(d_1)/S\sigma\sqrt{\tau}$. Here's the graph for gamma:

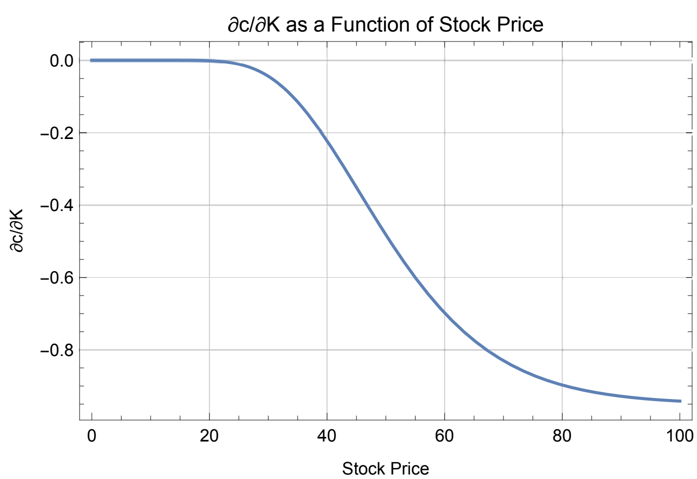
[2]One can intuitively infer the gamma for the call option visually by noting that gamma $= \dfrac{\partial^2 c}{\partial S^2}$; i.e., it measures how the hedge ratio changes as the price of the underlying varies. Gamma has a very small value for options which are either deeply out-of-the-money or deeply in-the-money (note that the slope is quite flat in these cases), but for options that are near-the-money, gamma is highly sensitive to changes in the price of the underlying asset.

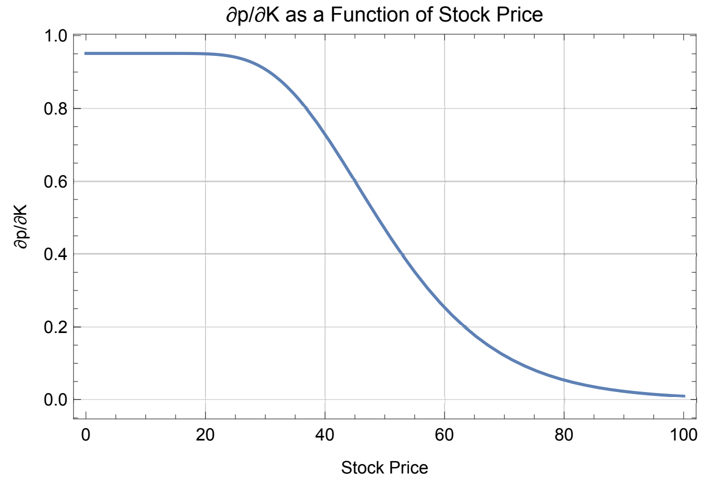
[3]The same point made concerning gamma for call options applies to put options; i.e., gamma has a very small value for options which are either deeply out-of-the-money or deeply in-the-money (note that the slope is quite flat in these cases), but for options that are near-the-money, gamma is highly sensitive to changes in the price of the underlying asset.



Here, gamma (for both the call and put) is maximized at a stock price slightly above \$40. As noted earlier in our analysis of delta, gamma has a very small value for options which are either deeply out-of-the-money or deeply in-the-money, but for options that are near the money, gamma is highly sensitive to changes in the price of the underlying asset.

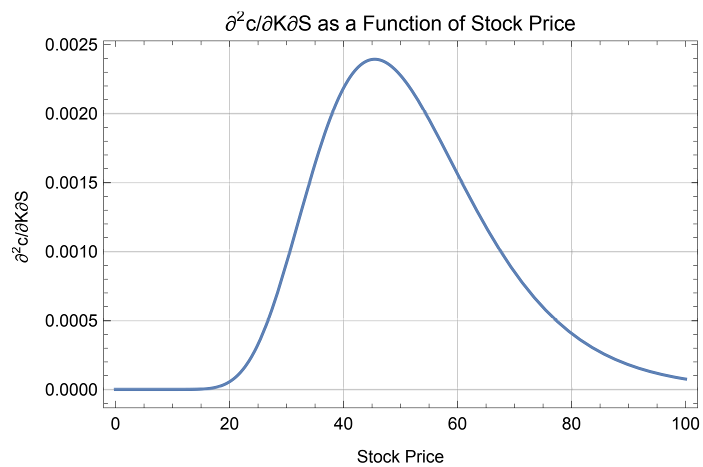
3.3 Relationship between $\frac{\partial c}{\partial K}$ and $\frac{\partial p}{\partial K}$ and the price of the underlying





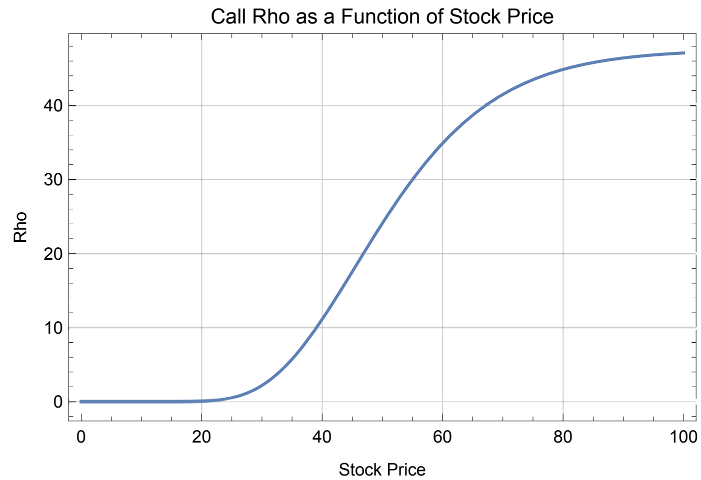
As in the case of the option gammas, we find that the sensitivity of the option price with respect to changes in the exercise price is particularly high for near-the-money options but rather insensitive in the case of out-of-the-money options.

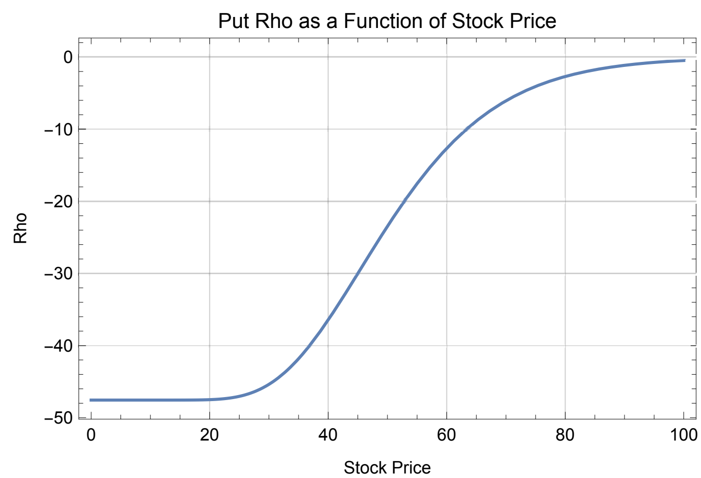
Since $\dfrac{\partial^2 c}{\partial K \partial S} = \dfrac{\partial^2 p}{\partial K \partial S} = e^{-r\tau} n(d_2)/S\sigma\sqrt{\tau}$, we can draw the graph for this "gamma" as follows:



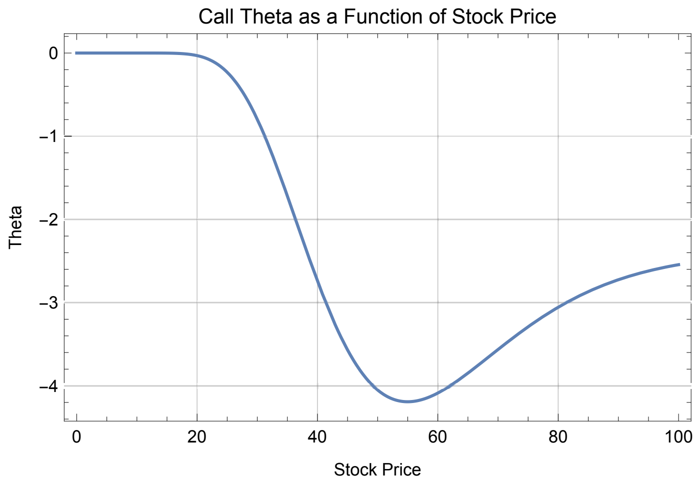
The graph for this "exercise price gamma" looks similar to gamma, but its value is maximized at a higher stock price.

3.4 Relationship between Option Rhos and the price of the underlying





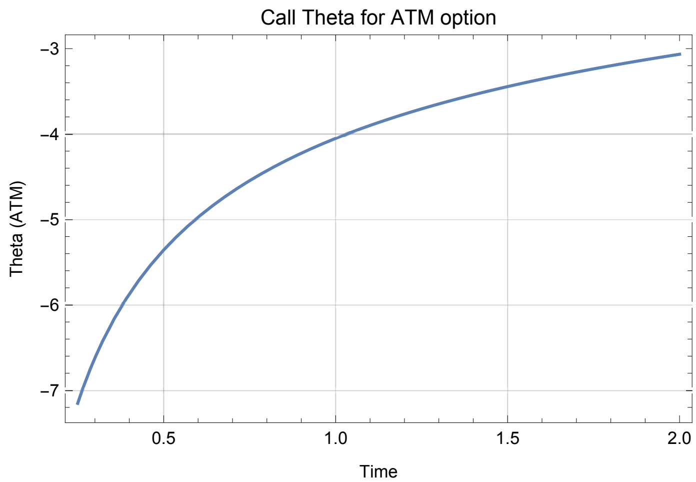
3.5 Relationship between Option Thetas and the price of the underlying



Theta is always negative for a call option. This is because, as time passes with all else remaining the same, the call option becomes less valuable. However, the rate at which this occurs depends upon the moneyness of the option. The variation of theta with stock price for a call option is shown above. When the stock price is very low, theta is close to zero. For an at-the-money call option, theta is large and negative. As the stock price becomes larger, theta converges in value toward $-re^{-r\tau}KN(d_2)$ (the other term vanishes since $n(d_1) \to 0$!

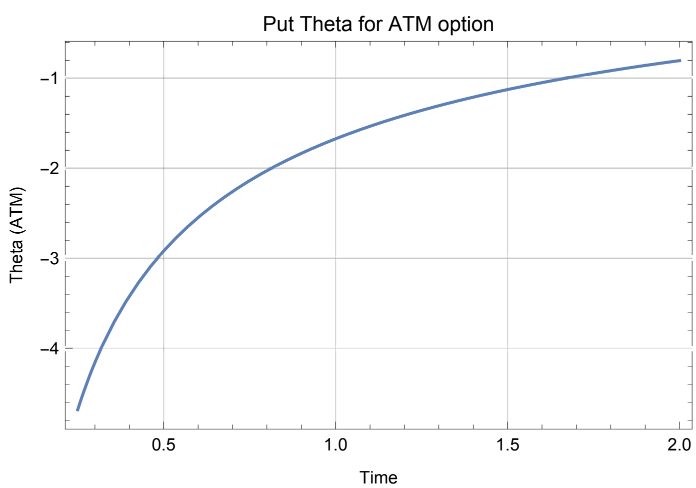
The pattern for the put theta is similar to the pattern for the call theta, only here we find that when the stock price is very low, theta is strongly positive. This because with the passage of time, the odds of a large move to a high price diminishes. However, as we go toward being at-the-money, theta turns negative, and converges toward zero as the stock price becomes very high.

Next, let's examine how the rate of time decay is related to the passage of time for an at-the-money call option:



The above graph shows that the call option's theta becomes more negative with the passage of time. Thus, the rate at which time decay occurs is faster for short-lived options than it is for longer lived options. Furthermore, the rate at which time decay changes with the passage of time is increasing, which implies that short-lived options (particularly options that are near or in-the-money) will lose value most rapidly as the expiration date draws close.

Next, let's examine how the rate of time decay is related to the passage of time for the put option; first consider at-the-money put (where the current stock price is $50):



The pattern for the at-the-money put theta is very similar to the pattern for the at-the-money call theta, only somewhat less attenuated. However, since the put theta is positive for deeply in-the-money puts, let's draw the same graph for an in-the-money put (where the current stock price is $30):

Here, the pattern for the in-the-money put theta is the reverse of the at-the-money put theta. Specifically, rather than decay with the passage of time, the in-the-money put theta expands with the passage of time. The reason for this that as the time to maturity decreases, this makes it much more likely that the put will end up expiring in-the-money.

3.6 Relationship between Option Vegas and the price of the underlying