Calculus and Optimization

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1 Introduction and Motivation

In Finance 4366, competency with basic mathematics and statistics principles is essential. The Finance 4366 math tutorial is designed to ensure competency with math principles that are needed for understanding how to identify, evaluate, price, and manage risk from individual and organizational perspectives. The 2-part statistics tutorial (to be taken up during the second week of class) is designed to ensure competency with stat principles used throughout the course.

In Section 2, we explain general principles behind calculus and optimization.[1] In Section 3, we delve more deeply into the logical framework behind calculus by showcasing some specific examples of how slopes or derivatives of functions are determined by applying the "rise over run" principle.

2 Calculus and Optimization: General Principles

An important part of the analytic toolkit for quantitatively-oriented academic disciplines such as economics and finance is basic calculus. During the late 19^{th} and early 20^{th} centuries, Alfred Marshall developed many important principles of economics by applying "marginal" analysis to topics such as profit maximization, utility maximization, and supply and demand.

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[1] In this teaching note, we focus primarily on *differential* calculus, which is the branch of calculus that studies how functions change with respect to changes in variables. For example, suppose you are interested in determining how the function $y = f(x)$ changes as x changes. The derivative of y with respect to x addresses this question by indicating the value of the function's slope at a given value of x.

Marginal analysis refers to the examination of how incremental, or marginal, changes in variables affect outcomes of decisions. For example, in economics, how does producing or consuming more or less goods and services affect firm profit and consumer welfare? In a similar vein, in finance and risk management, how does taking more or less risk affect investor welfare? Furthermore, are there any such things as "optimal" production, consumption, financial, or risk management decisions? By evaluating the marginal costs and benefits of decisions, individuals and organizations are better able to make rational and informed choices.

Marginal analysis involves the application of basic algebra and calculus principles. Algebra and calculus share in common the problem of calculating slope values, which represent rates of change in variables. The primary difference is that algebra indicates slope values for *discrete* changes, whereas calculus indicates slope values for *continuous* changes.

By a discrete change, we mean a change that is "countable"; e.g., in the equation $y = 10 + 5x$, a discrete change (Δy) occurs in y whenever a discrete change (Δx) occurs in x. For example, suppose the initial value for x is 2, and then we change x's value to 4. When $\Delta x = 2$, y changes from 20 to 30, so $\Delta y = 10$ and $\Delta y/\Delta x = 10/2 = 5$.

By continuous change, we mean a change that is infinitesimally small; specifically, we study how an infinitesimally small Δx (called dx) causes an infinitesimally small Δy (called dy) to occur. It turns out that in the case of our line example, $dy/dx = \Delta y/\Delta x = 5$. In cases involving nonlinear functions, such as a parabola (e.g., where $y = x^2$), $\Delta y/\Delta x$ is at best a crude approximation for dy/dx. However, as $\Delta x \to 0$, $\Delta y/\Delta x \to dy/dx$; in other words, the algebra result converges upon the calculus result.

In calculus, slope values are determined via a procedure called "differentiation", and the slope of a function such as a line or a curve is referred to as its "first derivative". The first derivative indicates the rate at which one variable (let's call it y) changes with respect to small changes in another variable (let's call the other variable x).

The second derivative corresponds to the rate of change in the slope itself. The second derivative comes in handy whenever we try to maximize or minimize a function. For example, suppose you are interested in determining how many units of a product to produce. If you produce and sell Q units of this product at a price of P dollars per unit, then total revenue is $TR = PQ$. If your total costs (TC) are fixed, then you would want to maximize total revenue, since this would maximize profit. However, one also typically incurs variable costs which increase as more units are produced, so in order to maximize profit, you'll want to produce up to the point at which the revenue generated from selling the last unit of product (also known as marginal revenue, or MR) is equal to the cost incurred from producing that unit (also known as marginal cost, or MC). Since marginal profit $MP = MR$ - $MC = 0$, this implies that total profit will be maximized when $MR = MC$.

Suppose total profit is $\pi = TR - TC$, where $TR = 30Q$ and $TC = 40 + 3Q^2$; then marginal profit $MP = d\pi/dQ = dTR/dQ - dTC/dQ = 30 - 6Q$, which equals 0 if $Q = 5$. We know that this is maximum profit because the 2^{nd} derivative (which corresponds to the slope of the slope of the total profit equation, or equivalently, the slope of the marginal profit equation) is negative; i.e., $d^2\pi/dQ^2 = dMP/dQ = -6$. Conceptually, a negative second derivative implies that as one moves away from $Q = 5$; e.g., by selecting either $Q = 4$ or $Q = 6$, then profit must be lower than at $Q = 5$. Note that at $Q = 5$, $\pi = 30(5) - 40 - 3(5^2) = 150 - 115 =$ \$35. Profit at $Q = 4$ is only $\pi = 30(4) - 40 - 3(4^2) = 120 - 88 =$ \$32, so producing 5 instead of 4 units results in \$3 more profit. Similarly, profit is also \$3 lower at $Q = 6$ compared with $Q = 5$; note that if $Q = 6$, then $\pi = 30(6) - 40 - 3(6^2) = 180 -$ \$148 = \$32. Finally, also note that at $Q = 5$, $MR = 30$ and $MC = 6(5) = 30$.

In summary, the total profit function is $\pi = \pi(Q)$; i.e., profit is a function of only one variable, Q. Thus, we maximize π by determining marginal profit (accomplished by differentiating π with respect to Q), setting marginal profit equal to zero, and solving for $Q = Q^*$. We know that Q^* is the profit-maximizing quantity because the second derivative of π with respect to Q, $d^2\pi/dQ^2 < 0$ for all possible values of Q.

3 Specific Calculus Examples

3.1 Slope of a linear function ("constant" rule of calculus)

Suppose we wish to determine the slope of the linear function given by $y = f(x) = a + bx$, as depicted in Figure 1:

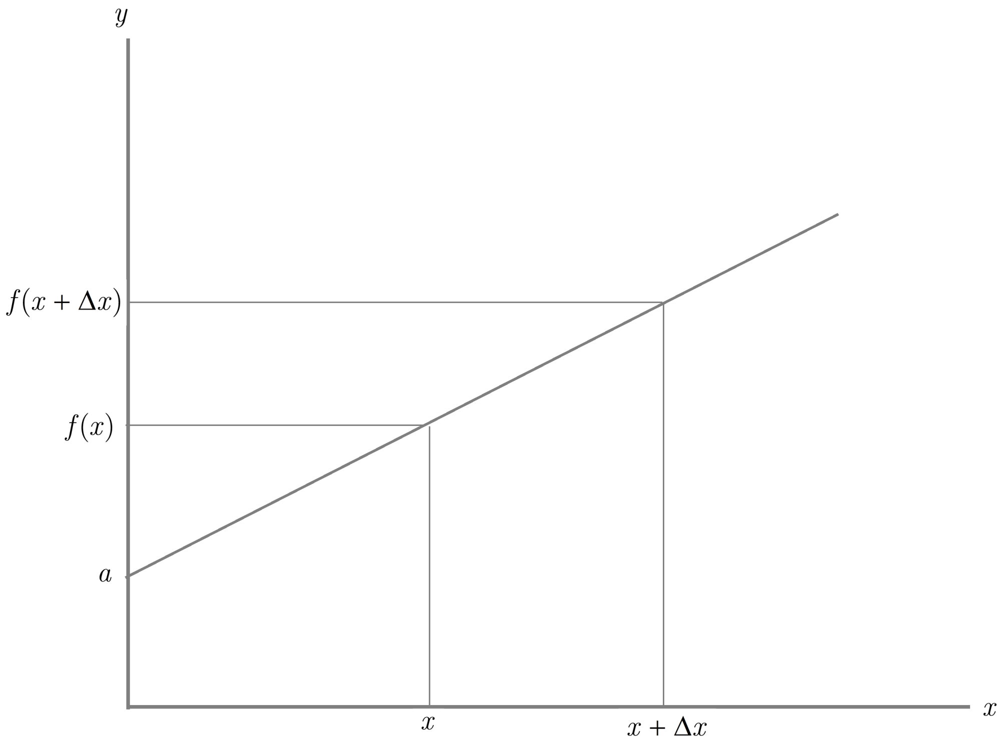

Figure 1: Graph of $y = f(x) = a + bx$.

As noted earlier, the general rule for calculating math derivatives involves applying the "rise-over-run" principle. Since $y = f(x)$, a Δx change in x (i.e., "run") from x to $x+\Delta x$ produces a corresponding change in y (i.e., "rise") from $f(x)$ to $f(x + \Delta x)$. Thus, the derivative of y with respect to x, i.e., $\dfrac{dy}{dx}$ is calculated as follows:

$$\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

In the case of the linear function $y = f(x) = a + bx$,

$$\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \left[\frac{(a + bx + b\Delta x) - (a + bx)}{\Delta x} \right] = \frac{b\Delta x}{\Delta x} = b.$$

Thus, the slope of a linear function such as $y = f(x) = a + bx$ is the coefficient b which is multiplied by the variable x. Thus, the slope value b is constant for all values of x.

3.2 Slopes of non-linear functions

3.2.1 Slope of a parabola

Now suppose the equation for the function $y = f(x)$ is non-linear; in this case, a parabola; i.e., $y = x^2$. Then

$$\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \left[\frac{(x + \Delta x)^2 - x^2}{\Delta x} \right].$$

We start by expanding the numerator of the ratio which appears on its right-hand side of the equation shown above:

$$\begin{aligned} (x + \Delta x)^2 - x^2 &= x^2 + \Delta x^2 + 2x\Delta x - x^2 \\ &= \Delta x^2 + 2x\Delta x. \end{aligned}$$

Dividing $\Delta x^2 + 2x\Delta x$ by Δx, we obtain

$$\lim_{\Delta x \to 0} \left[\frac{\Delta x^2 + 2x\Delta x}{\Delta x} \right] = \lim_{\Delta x \to 0} (\Delta x + 2x) = 2x.$$

Thus, unlike the result we obtained for our linear equation (where the slope value is constant for all values of x), here we find that the parabola's slope depends on the particular numerical value assumed by the x variable; e.g., if $x = 0$, then $f'(0) = 2(0) = 0$, if $x = 2$, then $f'(2) = 2(2) = 4$, if $x = 4$, then $f'(4) = 2(4) = 8$, and so forth.

3.2.2 Slope of a power function ("power" rule of calculus)

Power functions are functions in the form of $y = f(x) = kx^n$, where k is a nonzero coefficient, and n is a real number; note that the parabolic function considered earlier is a power function, where $k = 1$ and $n = 2$.

In the parabola example, we determined that if $y = x^2$, then $\frac{dy}{dx} = 2x^{2-1} = 2x$. In the general case where $y = x^n$, then $\frac{dy}{dx} = nx^{n-1}$. This is commonly referred to as the "power rule" of calculus.

3.3 Partial Differentiation

In sections 3.1 and 3.2 of this teaching note, we considered calculus principles for "*univariate*" cases, where y is a function of only one variable. A somewhat more interesting problem involves the so-called *multivariate* case, where the value of a function depends on two or more variables.

Suppose $z = f(x,\ y)$. Then the *partial* derivative of the function, f, with respect to x (denoted by $\partial f/\partial x$) is:

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \left[\frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} \right],$$

and the partial derivative of f with respect to y (denoted by $\partial f/\partial y$) is:

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \left[\frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} \right],$$

Here, $\partial f/\partial x$ is the derivative of f with respect to x while holding y constant, whereas $\partial f/\partial y$ is the derivative of f with respect to y while holding x constant.

Numerical Example of Partial Differentiation: Suppose $z = f(x,\ y) = 2x^2 - 3x^2y + 5y + 1$. Then $\partial f/\partial x$= $4x - 6xy$ and $\partial f/\partial y$=-3x^2 + 5.

3.4 General Rules of Differentiation

Although we only showed the foundations for the constant and power rules in sections 3.1 and 3.2.2, the same rise-over-run logic used in those cases applies also to deriving other rules of differentiation, including the sum/difference rule, the product rule, the quotient rule, the chain rule, the exponential rule, and the logarithmic rule, all of which are summarized below:

1. **Constant Rule:** The derivative of a constant function is always zero. If $f(x) = c$ (where c is a constant), then $f'(x) = 0$.

2. **Power Rule:** The derivative of a function raised to a constant power is obtained by multiplying the constant power with the function raised to one less power. If $f(x) = x^n$ (where n is a constant), then $f'(x) = nx^{n-1}$.

3. **Sum/Difference Rule:** The derivative of the sum or difference of two functions is equal to the sum or difference of their individual derivatives. If $f(x) = u(x) + v(x)$, then $f'(x) = u'(x) + v'(x)$. Similarly, if $f(x) = u(x) - v(x)$, then $f'(x) = u'(x) - v'(x)$.

4. **Product Rule:** The derivative of a product of two functions is obtained by differentiating one function and multiplying it by the other function, then differentiating the second function and multiplying it by the first function, and finally adding the two products. If $f(x) = u(x) \cdot v(x)$, then $f'(x) = u'(x) \cdot v(x) + u(x) \cdot v'(x)$.

5. **Quotient Rule:** The derivative of the quotient of two functions is calculated by taking the denominator function squared and multiplying it by the derivative of the numerator function, then subtracting the numerator function multiplied by the derivative of the denominator function, all divided by the denominator function squared. If $f(x) = \dfrac{u(x)}{v(x)}$, then $f'(x) = \dfrac{u'(x) \cdot v(x) - u(x) \cdot v'(x)}{v(x)^2}$.

6. **Chain Rule:** The chain rule is used to find the derivative of a composite function. If $f(x) = g(h(x))$, where g is the outer function and h is the inner function, then $f'(x) = g'(h(x)) \cdot h'(x)$.

7. **Exponential and Logarithmic Function Rules**

$$\frac{d}{dx}(e^x) = e^x \quad \text{(where } e \text{ is the base of the natural logarithm)}$$

$$\frac{d}{dx}(\ln(x)) = \frac{1}{x} \quad \text{(natural logarithm)}$$