Geometric Brownian Motion, Itô's Lemma, and Risk Neutral Valuation

by James R. Garven*
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We begin our analysis by assuming that the price of an underlying asset (currently worth \$$S$) changes continuously over time according to the Geometric Brownian Motion equation; i.e,

$$dS = \mu S dt + \sigma S dz, \tag{1}$$

In equation (1), dS corresponds to the instantaneous change in the price of the asset; dS consists of a non-stochastic component (given by $\mu S dt$) and a stochastic component (given by $\sigma S dz$).

Itô's Lemma provides a method for determining the corresponding differential equation for the price of virtually any derivative security which derives its value from S. Suppose that $f = f(S,t)$ represents the price of some such derivative security. Since f is twice differentiable in S and once differentiable in t, Itô's Lemma justifies the use of the following Taylor-series-like expansion for the instantaneous change in the price of the derivative security (given by df):

$$df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial S} dS + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} dS^2. \tag{2}$$

Next, we simplify the right hand side of equation (2). Since the third term in that equation is a function of dS^2, we square the right hand side of equation (1) and obtain $dS^2 = (\mu S dt + \sigma S dz)^2 = \mu^2 S^2 dt^2 + \sigma^2 S^2 dt + 2\mu\sigma S dt^{3/2}$. However, since the first and third terms of this equation involve dt raised to powers greater than 1, this implies that $dS^2 = \sigma^2 S^2 dt$.[1]

*James R. Garven is the Frank S. Groner Memorial Chair in Finance and Professor of Finance & Insurance at Baylor University (Address: Foster 320.39, One Bear Place #98004, Waco, TX 76798, telephone: 254-710-6207, e-mail: James_Garven@baylor.edu).

[1] Pennacchi (2008) makes the following very important observation about Wiener processes (cf. footnote 12 on page 240): "...it may be helpful to remember that in the continuous-time limit $dz^2 = dt$, but $dzdt = 0$, and $dt^n = 0$ for $n > 1$."

Substituting this expression for dS^2 into equation (2) yields equation (3):

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial S}dS + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}dt. \tag{3}$$

Suppose that at time t, we construct a hedge portfolio consisting of one unit of the derivative security worth $\$f(S,t)$ and a short position in some quantity Δ_t of the underlying asset worth $\$\Delta_t S_t$ per unit.[2] We express the hedge ratio Δ_t as a function of t because the portfolio will be *dynamically hedged*; i.e., as the price of the underlying asset changes through time, so will Δ_t.[3] Then the value of this hedge portfolio is $V_t = f(S,t) - \Delta_t S_t$, which implies

$$dV = df - \Delta_t dS = \underbrace{\left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}\right) dt}_{\text{deterministic}} + \underbrace{\left(\frac{\partial f}{\partial S} - \Delta_t\right) dS}_{\text{stochastic}}. \tag{4}$$

Note that there are stochastic as well as deterministic components on the right-hand side of equation (4). The deterministic component is represented by the first product involving dt, whereas the stochastic component is represented by the second product involving dS. However, by setting Δ_t equal to $\frac{\partial f}{\partial S}$, the stochastic component disappears since $\frac{\partial f}{\partial S} - \Delta_t = 0$, leaving:

$$dV = df - \Delta_t dS = \left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}\right) dt. \tag{5}$$

Since this is a perfectly hedged portfolio, it has no risk. In order to prevent arbitrage, the hedge portfolio must earn the riskless rate of interest r; i.e.,

$$dV = rV dt. \tag{6}$$

We will assume that $\Delta_t = \frac{\partial f}{\partial S}$, so $V = f(S,\ t)$ - $\frac{\partial f}{\partial S}S$. Substituting this into the right-hand side of equation (6) and equating the result with the right-hand side of equation (5), we

[2]Without loss of generality, this hedge portfolio can also consist of a *short* position in the derivative security and a *long* position in the underlying asset; such is the approach taken in Hull's derivation of the Black-Scholes-Merton differential equation (see pp. 331-332 of Hull (9th edition)).

[3]Intuitively, as the underlying asset price increases (decreases), then Δ_t must also increase (decrease) in order to form a perfect hedge, since changes in the price of the derivative security will more (less) closely mimic changes in the underlying asset price as it increases (decreases).

obtain:

$$r\left(f - S\frac{\partial f}{\partial S}\right)dt = \left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}\right)dt. \tag{7}$$

Dividing both sides of equation (7) by dt and rearranging results in the Black-Scholes-Merton (non-stochastic) partial differential equation:

$$rf = \frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}. \tag{8}$$

Equation (8) shows that the valuation relationship between a derivative security and its underlying asset is deterministic because dynamic hedging enables the investor to be perfectly hedged over infinitesimally small units of time. Since risk preferences play no role in this equation, this implies that the price of a derivative security may be calculated *as if* investors are risk neutral, in the sense that the expected rate of return on the underlying asset is set equal to the riskless rate of interest.[4] As Hull points out in the sixth section of his "The Black-Scholes-Merton Model" chapter, equation (8) provides the necessary framework for obtaining arbitrage-free prices for derivative securities that reference S as the underlying asset. For example, in the case of a call option, one can determine f's value by solving equation (8) subject to the "key" boundary condition that $f = \max(S - K, 0)$ at time T.[5] Furthermore, one can also confirm whether any particular derivative pricing formula correctly indicates the arbitrage-free price by showing that it satisfies equation (8). If equation (8) is not satisfied, then the price indicated by such a formula cannot possibly be arbitrage-free.

It should be noted that the partial derivatives given by $\frac{\partial f}{\partial t}$, $\frac{\partial f}{\partial S}$, and $\frac{\partial^2 f}{\partial S^2}$ in equation (8) correspond to the so-called "Greeks" for pricing derivative securities. The partial derivative of f with respect to t ($\frac{\partial f}{\partial t}$) is called the derivative's "theta". Theta measures how the price of the derivative changes with respect to the passage of time.[6] The partial derivative of

[4] Of course, this same result (i.e., risk neutral valuation) also obtains under the discrete-time case using the binomial model.

[5] The pricing equations for call and put options can also be determined via integration; e.g., see Derivation and Comparative Statics of the Black-Scholes Call and Put Option Pricing Equations.

[6] For call options and forward contracts, theta is negative for all possible values of the underlying asset; i.e., as the time to expiration or maturity of the derivatives contract becomes smaller, the value of the derivative declines (holding other factors constant). On the other hand, theta is positive for deeply in-the-money and negative for out-of-the-money put options.

f with respect to S ($\frac{\partial f}{\partial S}$) is called the derivative's "delta". Delta indicates the number of units of the underlying asset per unit of the derivative security which must be held so as to render the hedge portfolio riskless. Finally, the second partial derivative of f with respect to S ($\frac{\partial^2 f}{\partial S^2}$) is called the derivative's "gamma". Gamma measures the rate of change in the delta with respect to changes in the price of the underlying asset.

We conclude this teaching note by using risk neutral valuation to determine the value of a forward contract and confirming that the risk neutral valuation formula satisfies the Black-Scholes-Merton equation (8). Consider a long forward contract that matures at time T and has a delivery price of K. At maturity, the value of this contract is:

$$S_T - K. \tag{9}$$

Applying risk neutral valuation, we can find the date t value (f) of this forward contract by discounting the risk neutral expected value of its maturity value at the riskless rate of interest:

$$f = e^{-r(T-t)}\hat{E}(S_T - K) = e^{-r(T-t)}\hat{E}(S_T) - Ke^{-r(T-t)}, \tag{10}$$

where $\hat{E}(S_T - K)$ represents the risk neutral expected value of the value of the forward contract at maturity. Since $\hat{E}(S_T) = Se^{r(T-t)}$, equation (10) may be rewritten as:

$$f = S - Ke^{-r(T-t)}. \tag{11}$$

Next, we confirm that equation (11) satisfies equation (8). In order to do this, we need to determine the values of the partial derivatives for $\frac{\partial f}{\partial t}$, $\frac{\partial f}{\partial S}$, and $\frac{\partial^2 f}{\partial S^2}$ from equation (11). Since $\frac{\partial f}{\partial t} = -rKe^{-r(T-t)}$, $\frac{\partial f}{\partial S} = 1$, and $\frac{\partial^2 f}{\partial S^2} = 0$, substituting these values into the right-hand side of equation (8) produces the following equation:

$$rf = -rKe^{-r(T-t)} + rS = r(S - Ke^{-r(T-t)}) \tag{12}$$

Thus, the value of the forward contract given equation (11) does indeed satisfy the Black-Scholes-Merton equation shown in equation (8).

References

Pennacchi, G. G. (2008): *Theory of Asset Pricing*. Pearson/Addison-Wesley Boston.