Geometric Brownian Motion Simulations

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Starting from the assumption that returns on speculative assets are Markovian; i.e., that they represent serially uncorrelated "random walks", it follows that returns conform to *stochastic* processes which are partially deterministic and partially random in nature. The deterministic component of the stochastic process is captured by the so-called "drift", or "trend" rate ($\mu\delta t$ in the discrete case and μdt in the continuous case, both of which measure the average change in the random variable per time unit), whereas the random component is captured by the so-called variance rate (which measures the extent to which returns deviate per unit time from the time trend). The variance rate calculation is based on the assumption that over each δt or dt time interval, a standard normal shock (indicated by the Greek letter epsilon (ε)) occurs. Since ε is a standard normal random variable, $E(\varepsilon) = 0$ and $\sigma_\varepsilon = 1$. These assumptions imply that discrete-time returns conform to the following *difference* equation:

$$\delta S/S = \mu\delta t + \sigma\varepsilon\sqrt{\delta t}. \tag{1}$$

If we allow the length of the timestep δt in equation (1) to become arbitrarily small; i.e., let $\delta t \to 0$, then equation (1) can be written as a *differential* equation:

$$dS/S = \mu dt + \sigma\varepsilon\sqrt{dt}. \tag{2}$$

Typically, the second term on the right-hand sides of both of these equations is written as $\sigma\delta z$ and σdz respectively. The δz and dz terms are commonly referred to as Wiener or Brownian motion processes. The Wiener process has been used in physics to describe the motion of a particle that is subject to a large number of small and random (standard normal) molecular shocks. Similarly, in finance we use the Wiener process to describe random variations in returns which occur over discrete as well as continuous time intervals. The role of σ in these equations is to scale the overall *amplitude* of the shocks, so if you have an asset which has a relatively large σ compared with another otherwise identical asset with a smaller σ, then the deviations from the average return over time will be larger for the process which has the higher σ; e.g., compare Figures 1 and 2:

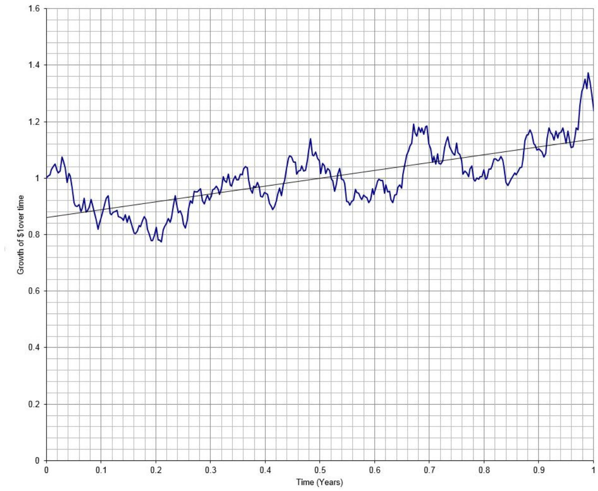

Figure 1. Simulation of the growth of $1 using daily data; $\mu = .10$, $\sigma = .40$.

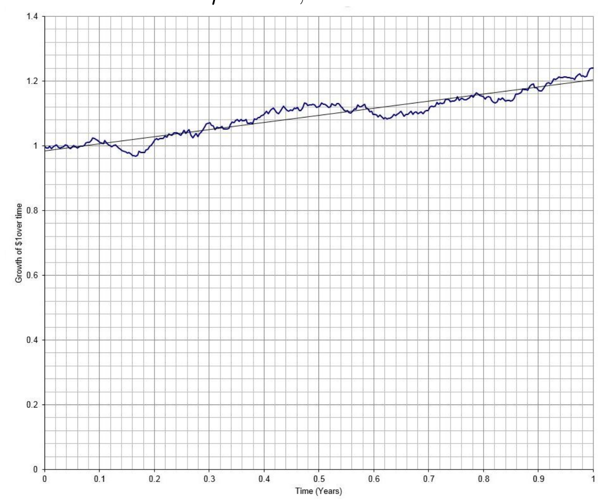

Figure 2. Simulation of the growth of $1 using daily data; $\mu = .10$, $\sigma = .10$.

Equations (1) and (2) are commonly referred to as Geometric Brownian Motion equations. Equation (2) is widely used in continuous time finance theory. Assuming the underlying asset return evolves according to equation (2), then Itô's Lemma makes it possible to infer the stochastic process and related probability distribution for virtually *any* kind of financial derivative (i.e., not just options and futures).