1-timestep Delta Hedging, Replicating Portfolio, and Risk Neutral Valuation Class Problems

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Problem 1: Use the following parameter values to price a call and put option by applying the Delta Hedging, Replicating Portfolio, and Risk Neutral Valuation approaches:

- S = current price of (non-dividend paying) underlying asset = \$50;
- K = exercise price = \$50;
- r = annualized riskless rate of interest = 3%;
- σ = annualized volatility (standard deviation of return) for the underlying asset = .1;
- δt = length of time-step in years = 1;
- u = one plus the rate of return on the underlying asset after one up move = $e^{\sigma\sqrt{\delta t}} = e^{.1\sqrt{1}} = 1.1052$; and
- d = one plus the rate of return on the underlying asset after one down move = $e^{-\sigma\sqrt{\delta t}} = e^{-.1\sqrt{1}} = .9048$.

1 Delta Hedging Approach for Arbitrage-Free Call Price

We start by creating a riskless bond by forming a perfectly hedged portfolio consisting of a long position in a call option and short position in Δ shares of the underlying asset.[1] The current value of this portfolio is

$$V_H = C - \Delta S = C - \Delta 50. \tag{1}$$

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[1] As we discussed in class, this particular trading strategy synthetically replicates a borrowing transaction; a lending transaction can be modeled by selling the call and buying Δ shares of the underlying asset. Showing this is left as an exercise for the reader.

At node u, the value of the hedge portfolio is equal to $V_H^u = C_u - \Delta uS$, and at node d, the value of the hedge portfolio is equal to $V_H^d = C_d - \Delta dS$. Since $uS = 50(1.1052) = \$55.26$ and $dS = 50(.9048) = \$45.24$, it follows that $V_H^u = 5.26 - \Delta 55.26$ and $V_H^d = 0 - \Delta 45.24$. Suppose we select Δ such that the hedge portfolio is riskless; i.e., $V_H^u = V_H^d$. Solving for Δ, we obtain:

$$V_H^u = V_H^d \Rightarrow 50(1.1052) = \$55.26 = -\Delta 45.24 \Rightarrow \Delta = .525. \tag{2}$$

Substituting $\Delta = .525$ into our expressions for V_H^u and V_H^d, we obtain $V_H^u = V_H^d = -\$23.75$. Thus, the value of a riskless hedge portfolio consisting of one call option and a short position in .525 shares of stock is equivalent in value to a *short* position in a riskless bond; i.e., it is *as if* we have created a riskless loan in which we borrow a principal value of \$23.05 up front and pay back principal (\$23.05) plus interest (\$0.75) in one year. In order to prevent arbitrage, the current value of the hedge portfolio, $V_H = C - \Delta 50 = C - .525(50) = C - 26.25$, must be equal to the present value of our short bond position, which is $e^{-r\delta t} V_H^d = -e^{-.03}(23.75) = -\$23.05$; consequently, $C = \$3.20$.

Since we now have the arbitrage-free price for the call option, we can use the put-call parity equation to find the arbitrage-free price of an otherwise identical put option. The put-call parity equation is given by equation (3):

$$C + Ke^{-r\delta t} = P + S. \tag{3}$$

Thus,

$$P = C + Ke^{-r\delta t} - S = \$3.2 + \$50e^{-.03} - \$50 = \$1.72. \tag{4}$$

We can also determine the arbitrage-free price for the put option via the delta hedging approach. Since the prices of a put option and its underlying stock are inversely related, we form a hedge portfolio consisting of a long position in one put option and a long position in

Δ shares of stock.[2] The current value of this portfolio is

$$V_H = P + \Delta S = P + \Delta 50. \tag{5}$$

At node u, the value of the hedge portfolio is equal to $V_H^u = P_u + \Delta uS$, and at node d, the value of the hedge portfolio is equal to $V_H^d = P_d + \Delta dS$. Since $uS = \$55.26$ and $dS = \$45.24$, it follows that $V_H^u = 0 + \Delta 55.26$ and $V_H^d = 4.76 + \Delta 45.24$. Suppose we select Δ such that the hedge portfolio is riskless; i.e., $V_H^u = V_H^d$. Solving for Δ, we obtain:

$$V_H^u = V_H^d \Rightarrow \Delta 55.26 = 4.76 + \Delta 45.24 \Rightarrow \Delta = .475. \tag{6}$$

Substituting $\Delta = .475$ into our expressions for V_H^u and V_H^d, we obtain $V_H^u = V_H^d = \$26.25$. Thus, the value of a riskless hedge portfolio consisting of one put option and .475 shares of stock is equivalent in value to a *long* position in a riskless bond. In order to prevent arbitrage, the current value of this long bond position, $V_H = P + \Delta 50 = \$e^{-.03}26.25 = \$25.47$. In order to prevent arbitrage, the current value of the hedge portfolio, $V_H = P + \Delta 50 = P + .475(50) = P + 23.75 \Rightarrow P + \$23.75 = \$25.47 \Rightarrow P = \$1.72$.

2 Replicating Portfolio Approach in a Single Period

Next, we consider the replicating portfolio approach for determining the arbitrage-free prices of the call and put options. The current value of the replicating portfolio for the call option is $V_{RP}^C = \Delta_C S + B_C$, where $\Delta_C = \dfrac{C_u - C_d}{S(u-d)} = \dfrac{5.26}{50(.2003)} = .525$ and $B_C = \dfrac{uC_d - dC_u}{e^{r\delta t}(u-d)} = \dfrac{-.9048(5.26)}{e^{.03}(.2003)} = -\$23.05$. Thus, we can replicate the call option by purchasing .525 of a share for \$26.25 and borrowing \$23.05, which implies that $C = V_{RP}^C = 26.25 - 23.05 = \$3.20$. Similarly, the current value of the replicating portfolio for the put option is $V_{RP}^P = \Delta_P S + B_P$,

[2] This trading strategy synthetically replicates a *lending* transaction; a borrowing transaction can be modeled by selling the put and shorting Δ shares of the underlying asset. Showing this is left as an exercise for the reader.

where $\Delta_P = \dfrac{P_u - P_d}{S(u-d)} = \dfrac{-4.76}{50(.2003)} = -.475$ and $B_P = \dfrac{uP_d - dP_u}{e^{r\delta t}(u-d)} = \dfrac{1.1052(4.76)}{e^{.03}(.2003)} = \$25.47$. Thus, we can replicate the put option by shorting .475 of a share for \$23.75 and lending \$25.47, which implies that $P = V_{RP} = -23.75 + 25.47 = \$1.72$.

3 Risk Neutral Valuation Approach in a Single Period

Under risk neutral valuation, the pricing equation for a single time-step call option is $c = e^{-r\delta t}\left[qc_u + (1-q)\,c_d\right]$, and the pricing equation for a single time-step put option is $p = e^{-r\delta t}\left[qp_u + (1-q)\,p_d\right]$, where $q = \dfrac{e^{r\delta t} - d}{(u-d)}$. Thus, $q = \dfrac{e^{.03} - .9048}{(1.1052 - .9048)} = .627$, $c = e^{-.03}\left[.627(5.26)\right] = \$3.20$, and $p = e^{-.03}\left[.373(4.76)\right] = \$1.72$.