Actual versus Risk Neutral Probability of a Call Option Expiring in-the-money

by James R. Garven*
April 6, 2023

An important simplification for pricing an option in the discrete time binomial framework involves the use of the so-called "risk neutral" probability (q) of an up move. Since the expected value of the underlying asset price ($E(S_{\delta t})$) one δt timestep from today is $E(S_{\delta t}) = puS + (1-p)dS = e^{\mu \delta t}S$ (see pp. 20-21 of the Binomial Trees lecture note), it follows that the actual probability (p) of an up move is $p = (e^{\mu\delta t} - d)/(u-d)$, where μ corresponds to the annual (continuously compounded) expected rate of return on the underlying asset. We obtain the risk neutral probability q by replacing μ with the riskless rate of interest r:

$$q = (e^{r\delta t} - d)/(u-d). \tag{1}$$

Since $\mu - r > 0$ in a risk averse world, it follows that $p > q$.

As we segue from the discrete time framework over to the continuous time framework, we find that a similar relationship exists between the actual and risk neutral probabilities that a call option will expire in-the-money. Consider the Black-Scholes-Merton call option pricing equation:

$$C = SN(d_1) - Ke^{-rT}N(d_2), \tag{2}$$

where $d_1 = \dfrac{\ln(S/K) + (r + .5\sigma^2)T}{\sigma\sqrt{T}}$;

$d_2 = d_1 - \sigma\sqrt{T} = \dfrac{\ln(S/K) + (r - .5\sigma^2)T}{\sigma\sqrt{T}}$;

σ^2= variance of underlying asset's rate of return; and

$N(z)$ = standard normal distribution function evaluated at z.

*James R. Garven is the Frank S. Groner Memorial Chair in Finance and Professor of Finance & Insurance at Baylor University (Address: Foster 320.39, One Bear Place #98004, Waco, TX 76798, telephone: 254-307-1317, e-mail: James_Garven@baylor.edu).

In equation (2), $N(d_2)$ corresponds to the risk neutral probability that the call option expires in-the-money.[1]

Next, consider the calculation of the actual probability of a call option expiring in-the-money that is implied in the Wiener Processes and Îto's Lemma lecture note. The Geometric Brownian Motion equation $dS = \mu S dt + \sigma S dz$ (equation (1) on page 7 of that note) implies that date T stock prices ($\tilde{S}_T$) are lognormally distributed; thus, they are bounded from below at 0, and unbounded from above. Since the gross return $\tilde{S}_T/S$ is determined by dividing $\tilde{S}_T$ by the current (known) stock price S, it follows that this ratio is also lognormally distributed. Furthermore, since the natural logarithm of a lognormally distributed random variable is normally distributed, it follows that the natural logarithm of the $\tilde{S}_T/S$ ratio, $\ln(\tilde{S}_T/S)$, is normally distributed and corresponds to the (continuously compounded) rate of return on the stock. By applying Îto's Lemma (see pp. 11-13 of the Wiener Processes and Îto's Lemma lecture note), we find that the continuously compounded rate of return on the stock is normally distributed with mean $(\mu - .5\sigma^2)T$ and variance $\sigma^2 T$; i.e., $\ln \tilde{S}_T/S \sim N((\mu - .5\sigma^2)T, \sigma^2 T)$.

Keeping these results in mind, we next calculate the actual probability that a call option will expire in-the-money. From equation (2), we know that the risk neutral probability this will occur is $N(d_2) = N\left(\frac{\ln(S/K) + (r - .5\sigma^2)T}{\sigma\sqrt{T}}\right)$. Thus, the actual probability of expiring in-the-money is calculated by replacing the riskless rate of return r with the expected return μ in the numerator for d_2; i.e., the actual probability this will occur is $N\left(\frac{\ln(S/K) + (\mu - .5\sigma^2)T}{\sigma\sqrt{T}}\right)$.[2]

[1] The mathematical details behind this insight are provided in Section 3 of Garven (2017).

[2] See the "Actual versus Risk Neutral Probabilities" spreadsheet for a numerical example of these calculations.

References

Garven, J. R. (2017): "Derivation and Comparative Statics of the Black-Scholes Call and Put Option Pricing Formulas," *unpublished manuscript*.