Calculating (Math) Derivatives

by James R. Garven*
January 23, 2022

In Finance 4366, competency with basic math and stat principles is essential. The Finance 4366 math tutorial is designed to ensure competency with math principles that are needed for understanding how to price financial derivatives (such as options and futures contracts) and designing risk management strategies using derivatives. The 2-part stat tutorial (to be taken up during the second week of class) is designed to ensure competency with stat principles which are used throughout the course.

In this teaching note, I briefly delve further into the logical foundations for calculating math derivatives.

1 Calculating Derivatives – finding slopes of functions

Suppose we wish to determine the slope of the function $y = f(x) = a + bx$ in Figure 1:

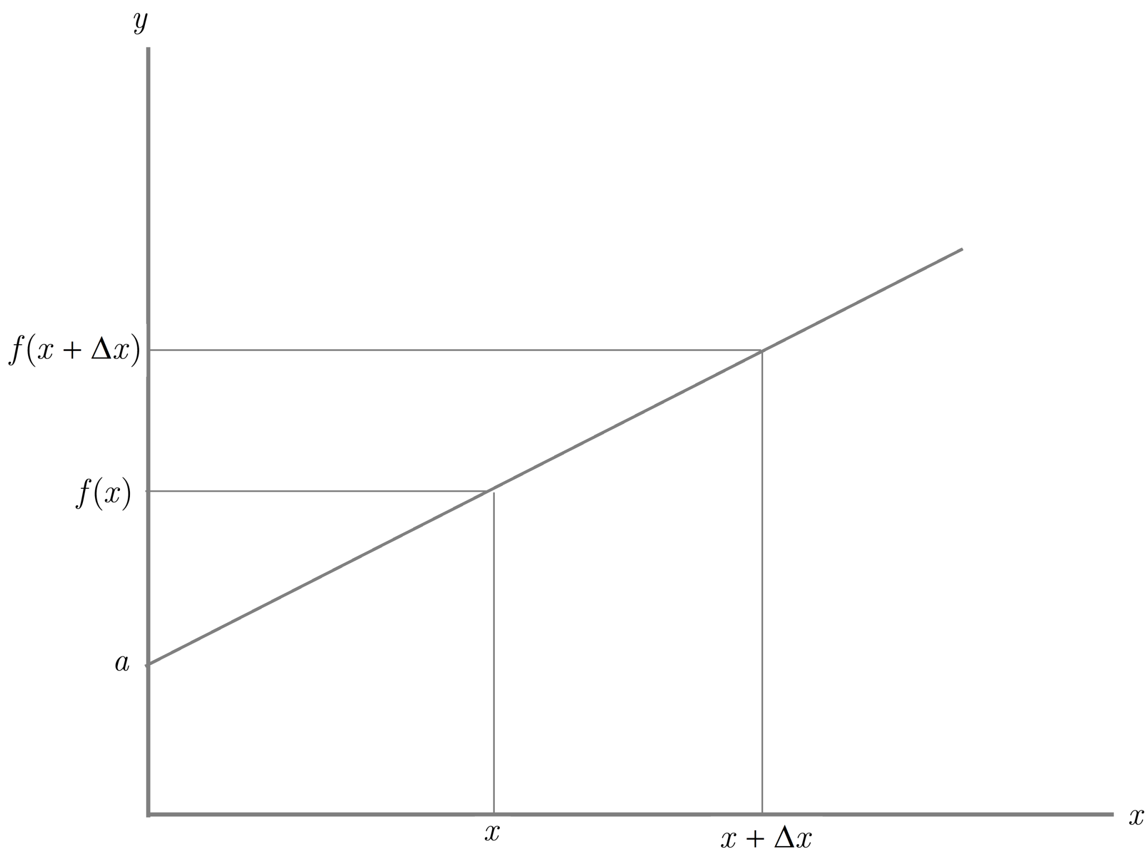

Figure 1: Determining slope for the function $y = f(x) = a + bx$.

*James R. Garven is the Frank S. Groner Memorial Chair in Finance and Professor of Finance & Insurance at Baylor University (Address: Foster 320.39, One Bear Place #98004, Waco, TX 76798, telephone: 254-307-1317, e-mail: James_Garven@baylor.edu).

Since the function $y = f(x) = a + bx$ is linear, its slope is constant for all values of x. We can determine the value of the slope b by applying the "rise over run" principle. Specifically, we can find b by solving the following equation:

$$b = \frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{\Delta y}{\Delta x}.$$

In this equation, it does not matter what the values are for x or Δx; its solution will always yield the number $b = \dfrac{\Delta y}{\Delta x}$.

Next, consider a non-linear (concave) function as shown in Figure 2:

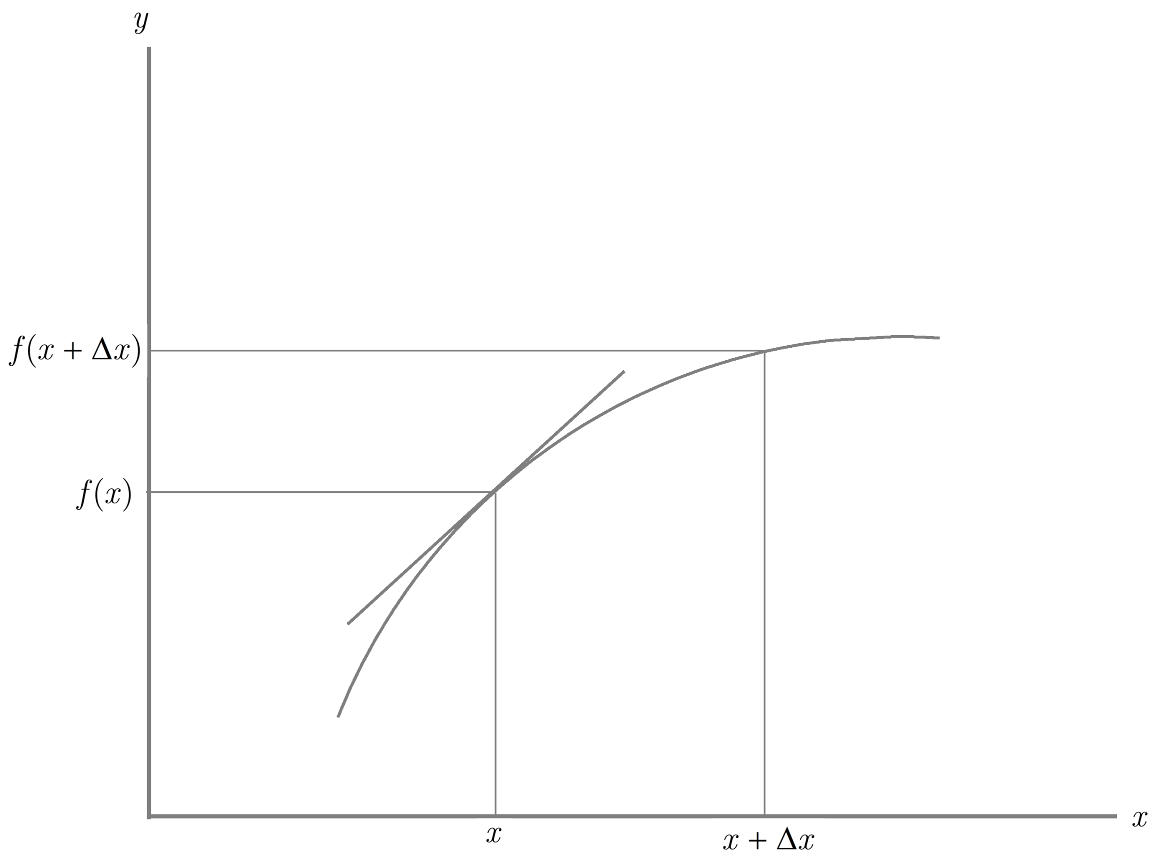

Figure 2: Determining slope of a nonlinear function at coordinates $(x, f(x))$.

Suppose we wish to know the slope of this function at the point with coordinates $(x, f(x))$. Since this is a concave function, its slope, as indicated by the tangent line segment at that point, is steeper than it is at the point with coordinates $(x + \Delta x, f(x + \Delta x))$. Although we can still apply the "rise over run" principle, to find the slope of the function at the point with coordinates $(x, f(x))$, we need to determine the *limit* of the $\dfrac{f(x + \Delta x) - f(x)}{\Delta x}$ ratio as Δx becomes arbitrarily close to zero; i.e.,

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{dy}{dx} = f'(x).$$

Here, $\frac{dy}{dx} = f'(x)$ corresponds to the derivative of y with respect to x, evaluated at the point with coordinates $(x, f(x))$.

2 Numerical Example (parabola):

Suppose the function $y = f(x)$ is a parabola; i.e., $y = x^2$. Then

$$\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \left[\frac{(x+\Delta x)^2 - x^2}{\Delta x} \right].$$

We'll solve this equation by expanding the numerator of the ratio which appears on its right-hand side:

$$\begin{aligned}(x+\Delta x)^2 - x^2 &= x^2 + \Delta x^2 + 2x\Delta x - x^2 \\ &= \Delta x^2 + 2x\Delta x.\end{aligned}$$

Dividing $\Delta x^2 + 2x\Delta x$ by Δx, we obtain

$$\lim_{\Delta x \to 0} \left[\frac{\Delta x^2 + 2x\Delta x}{\Delta x} \right] = \lim_{\Delta x \to 0} (\Delta x + 2x) = 2x.$$

Thus, the rate of change of the parabola depends upon the particular value of x; e.g., if $x = 0$, then $f'(0) = 2(0) = 0$, if $x = 2$, then $f'(2) = 2(2) = 4$, if $x = 4$, then $f'(4) = 2(4) = 8$, and so forth. Also, note that the procedure for determining the slope of a parabola represents a "special case" of the power rule. Specifically, suppose $y = x^n$. Then

$$\frac{dy}{dx} = nx^{n-1}.$$