Finance 4366 Dynamic Delta Hedging Numerical Example (calls and puts)

by James R. Garven*

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1 Introduction

In theory, the delta hedging approach to pricing options involves synthetically creating a riskless bond by forming a perfectly hedged portfolio consisting of either a call option or a put option and the underlying stock. In practice, delta hedging represents an options trading strategy which aims to reduce/hedge the risk associated with price movements in the underlying asset by using offsetting long or short positions in options.

In my "Binomial Option Pricing Model (single-period)" teaching note, I demonstrate delta hedging in a "static", one-period setting. Here, I introduce "dynamic" delta hedging by exploring how the passage of time affects this trading strategy. Basically, dynamic hedging implies that the investor will need to rebalance the hedge portfolio as the price of the underlying asset changes over time. For calls (puts), this means that as the underlying asset increases (decreases) in value, then the investor will need to increase exposure to the underlying asset so as to offset price movements in his/her option position.

2 Dynamic delta hedging: long stock, short call

The price of a share is currently \$100. It is known that at the end of 1 month, the share price will be either \$105 or \$95, and at the end of 2 months, the share price will be \$110.25, \$99.75, or \$90.25.[1] The riskless rate of interest is 5% per year, and the exercise price on a call option that expires in 2 months is \$100.

Next, we show how a portfolio consisting of Δ shares of stock and a short position in 1 call option can be made riskless at the very beginning of the binomial tree, and also at nodes u and d. We also show that such a portfolio is *guaranteed* to provide an annualized return of 5% at each of these nodes.

SOLUTION: The binomial tree for stock and call option prices looks like this:[2]

*James R. Garven is the Frank S. Groner Memorial Chair in Finance and Professor of Finance & Insurance at Baylor University (Address: Foster 320.39, One Bear Place #98004, Waco, TX 76798, telephone: 254-710-6207, e-mail: James_Garven@baylor.edu).

[1]These share prices imply that $u = 1.05$ and $d = .95$.

[2]The call option price tree is calculated by determining the payoff on the call option at nodes uu, ud,

		S_{uu} = \$110.25
		c_{uu} = \$10.25
	S_u = \$105	
	c_u = \$5.53	
S = \$100		S_{ud} = \$99.75
c = \$2.98		c_{ud} = \$0
	S_d = \$95	
	c_d = \$0	
		S_{dd} = \$90.25
		c_{dd} = \$0

- In order to construct a riskless hedge portfolio at the inception of the binomial tree, we must select a hedge ratio Δ such that the value of the hedge portfolio in the u state is equal to the value of the hedge portfolio in the d state; i.e., $V_H^u = V_H^d$. Since $V_H^u = \Delta S_u - c_u$ $= \Delta(105) - 5.53$ and $V_H^d = \Delta S_d - c_d = \Delta(95) - 0$, it follows that $\Delta = 5.53/10 = 0.553$. Thus at node u, the hedge portfolio is worth $V_H^u = (0.553)(105) - 5.53 =$ \$52.54, and at node d, the hedge portfolio is worth $V_H^d = (0.553)(\$95) = \$52.54$. At the very beginning of the tree, the hedge portfolio is worth $V_H = 0.553(\$100) - \$2.98 = \$52.32$. Thus, $V_H^u = V_H^d = e^{r(1/12)}V \Rightarrow r = 12\ln(V_H^d/V_H) = 12\ln(52.54/52.32) = 5\%$; i.e., the hedge portfolio is guaranteed to provide an annualized return of 5% during the course of the first 1 month timestep, irrespective of whether the stock price initially moves up or down.

- In order to construct a riskless hedge portfolio at node u, we must select a hedge ratio Δ_u such that the value of the hedge portfolio in the uu state is equal to the value of the hedge portfolio in the ud state; i.e., $\Delta_u(110.25) - 10.25 = \Delta_u(99.75) \Rightarrow \Delta_u = 10.25/10.50 = .9762$. Thus at node uu, the hedge portfolio is worth $V_H^{uu} = \Delta_u S_{uu} - c_{uu} = (.9762)(\$110.25)$ – \$10.25 = \$97.38, and at node ud, the hedge portfolio is worth $V_H^{ud} = \Delta_u S_{ud} - c_{ud} =$ (.9762)(\$99.75) – \$4.80 = \$97.38. At node u, the hedge portfolio is worth $V_H^u = \$.9762(105)$ - \$5.53 = \$96.97. Thus, $r = 12\ln(V_H^{ud}/V_H^u) = 12\ln(97.38/96.97) = 5\%$; i.e., the hedge portfolio is guaranteed to provide an annualized return of 5% during the course of the second 1 month timestep, after an initial "up" move and irrespective of whether the subsequent move is up or down.

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and dd and then solving (via backward induction, using risk neutral valuation) for call option prices at nodes u, d, and at the tree's inception. Since this call option has an exercise price of \$100, this implies that $c_{uu} = \max[0, S_{uu} - K] = \$10.25$, whereas $c_{ud} = \max[0, S_{ud} - K] = \$0$ and $c_{dd} = \max[0, S_{dd} - K] = \$0$. The risk neutral probability of an up move is $q = \frac{e^{r\delta t} - d}{u - d} = \frac{e^{.05/12} - .95}{1.05 - .95} = .5418$. Since $c_{ud} = c_{dd} = 0, c_d = e^{-r\delta t}[qc_{ud} + (1-q)c_{dd}] = 0$. Solving for $c_u, c_u = e^{-r\delta t}[qc_{uu} + (1-q)c_{ud}] = e^{-.05/12}[.5418(10.25)] = 5.53$. Thus, $c = e^{-r\delta t}[qc_u + (1-q)c_d] = e^{-.05/12}[.5418(5.53)] = 2.98$.

- Suppose that at one timestep prior to expiration, we find ourselves in the down state. Our option delta in this state is equal to 0%; since the option will expire out of the money in both the *ud* and *dd* states, this means that we need to eliminate our exposure to the underlying asset because the call option is out of the money in both of these states (and there is no longer a need to form a hedge portfolio).

3 Dynamic hedging: long stock, long put

Next, we show how a portfolio consisting of Δ shares of stock and a long position in 1 put option can be made riskless at the very beginning of the binomial tree, and also at nodes u and d. We also show that such a portfolio is guaranteed to provide an annualized return of 5% at each of these nodes.

SOLUTION: The binomial tree for stock and put option prices looks like this:[3]

		$S_{uu} = \$110.25$
		$p_{uu} = \$0$
	$S_u = \$105$	
	$p_u = \$0.11$	
$S = \$100$		$S_{ud} = \$99.75$
$p = \$2.15$		$p_{ud} = \$.25$
	$S_d = \$95$	
	$p_d = \$4.58$	
		$S_{dd} = \$90.25$
		$p_{dd} = \$9.75$

- In order to construct a riskless hedge portfolio at the inception of the binomial tree, we must select a hedge ratio Δ such that the value of the hedge portfolio in the u state is equal to the value of the hedge portfolio in the d state; i.e., $V_H^u = V_H^d$. Since $V_H^u = \Delta S_u + p_u$ $= \Delta(105) + .11$ and $V_H^d = \Delta S_d + p_d = \Delta(95) + 4.58$, it follows that $\Delta = 4.47/10 = 0.447$. Thus at node u, $V_H^u = (0.447)(105) + .11 = \$47.05$, and at node d, the hedge

[3]The put option price tree is calculated by determining the payoff on the put option at nodes uu, ud, and dd and then solving (via backward induction, using risk neutral valuation) for put option prices at nodes u, d, and at the tree's inception. Since put option has an exercise price of \$100, this implies that $p_{uu} = \max[0, K - S_{uu}] = \$0$, whereas $p_{ud} = \max[0, K - S_{ud}] = \$0.25$ and $p_{dd} = \max[0, K - S_{dd}] = \$9.75$. Solving for p_u and p_d, $p_u = e^{-r\delta t}[qp_{uu} + (1-q)p_{ud}] = e^{-.05/12}[.4582(0.25)] = 0.11$ and $p_d = e^{-r\delta t}[qp_{ud} + (1-q)p_{dd}] = e^{-.05/12}[.5418(0.25) + .4582(9.75)] = 4.58$. Thus, $p = e^{-r\delta t}[qp_u + (1-q)p_d] = e^{-.05/12}[.5418(0.11) + .4582(4.58)] = 2.15$.

portfolio is worth V_H^d= (0.447)(\$95)+4.58 = \$47.05. At the very beginning of the tree, the hedge portfolio is worth $V_H = 0.447(\$100) + \$2.15 = \$46.85$. Thus, $r = 12\ln(V_H^d/V_H) = 12\ln(47.05/46.85) = 5\%$; i.e., the hedge portfolio is guaranteed to provide an annualized return of 5% during the course of the first 1 month timestep, irrespective of whether the stock price initially moves up or down.

- In order to construct a riskless hedge portfolio at node u, we must select a hedge ratio Δ_u such that the value of the hedge portfolio in the uu state is equal to the value of the hedge portfolio in the ud state; i.e., $\Delta_u(110.25) + 0 = \Delta_u(99.75) + .25 \Rightarrow \Delta_u = .25/10.50 = .0238$. Thus at node uu, the hedge portfolio is worth $V_H^{uu}= \Delta_u S_{uu} + p_{uu}=$ (.0238)(\$110.25) +0 = \$2.62, and at node ud, the hedge portfolio is worth $V_H^{ud}= \Delta_u S_{ud} + p_{ud}=$ (.0238)(\$99.75)+\$.25 = \$2.62. At node u, the hedge portfolio is worth $V_H^u =$ \$.0238(105) + \$.11= \$2.62. Thus, $r = 12\ln(V_H^{ud}/V_H)= 12\ln(2.62/2.61) = 5\%$; i.e., the hedge portfolio is guaranteed to provide an annualized return of 5% during the course of the second 1 month timestep, after an initial "up" move and irrespective of whether the subsequent move is up or down.

- In order to construct a riskless hedge portfolio at node d, we must select a hedge ratio Δ_d such that the value of the hedge portfolio in the ud state is equal to the value of the hedge portfolio in the dd state; i.e., $V_H^{ud} = V_H^{dd} \Rightarrow \Delta_d(99.75)+.25 = \Delta_d(90.25)+9.75 \Rightarrow \Delta_d = 1$. Thus, by setting $\Delta_d = 1$, we offset price movements in the put option dollar for dollar with price movements in the underlying asset. At expiration, the hedge portfolio is worth \$100 in both the ud and dd states, and at node d, it is worth $V_H^d = 1(95) + 4.58 = \$99.58$. Thus, $r = 12\ln(V_H^{ud}/V_H)= 12\ln(100/99.58) = 5\%$; i.e., the hedge portfolio is guaranteed to provide an annualized return of 5% during the course of the second 1 month timestep, after an initial "down" move and irrespective of whether the subsequent move is up or down.