Finance 4366 Dynamic Replicating Portfolio Numerical Example (calls and puts)

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October 10, 2016

1 Introduction

The replicating portfolio approach to pricing options involves synthetically creating call and put options with combinations of the underlying asset and a riskless bond. One can replicate a call option by buying the underlying asset and shorting the bond, whereas a put option can be replicated by buying the bond and shorting the underlying asset. In a "static", single-period setting (see my "Binomial Option Pricing Model (single-period)" teaching note), an investor replicates the node u and d call and put option values by trading the replicating portfolio components in just the right proportions. The current market value of the replicating portfolio must equal the current market value of the option; otherwise investors can earn positive profits with zero risk and zero net investment by buying the less expensive investment and shorting the more expensive one. As in the case of the delta hedging approach, we invoke the no-arbitrage condition to establish that the price of the option must equal the value of its replicating portfolio.

Here, I introduce "dynamic" portfolio replication by exploring how the passage of time affects this trading strategy. Basically, dynamic portfolio replication implies that the investor must rebalance the underlying asset and bond portfolio components as the price of the underlying asset changes over time. For call options, this means that as the underlying asset increases (decreases) in value, then the investor must increase (decrease) exposure to the underlying asset while shorting more (less) of the riskless bond, so as to mimic how the value of the call option changes with the passage of time. Similarly, for puts, this means that as the underlying asset increases (decreases) in value, then the investor must short smaller (larger) dollar values of the underlying asset, coupled with a decrease (increase) in the amount being lent, so as to mimic how the value of the put option changes with the passage of time.

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2 Dynamic Call Option Replication

The price of a share is currently $S = \$100$. It is known that at the end of 1 month, the share price will be either $S_u = uS = \$105$ or $S_d = dS = \$95$, and at the end of 2 months, the share price will be $S_{uu} = uuS = \$110.25$, $S_{ud} = udS = \$99.75$, or $S_{dd} = ddS = \$90.25$.[1] The riskless rate of interest $r = 5\%$ per year, and the exercise price on a call option that expires in 2 months is $K = \$100$.

We dynamically replicate this call option via backward induction; i.e., we start at the end of the binomial tree and work our way back to the beginning. Suppose the initial price move for the underlying asset is "up". At node u, the investor forms a portfolio consisting of Δ_u shares of stock and $\$B_u$ in riskless bonds; this portfolio is designed to replicate the node uu and node ud values for the call option (given by c_{uu} and c_{ud}) described above. The cost of this node u "replicating" portfolio is $V_{RP}^u = \Delta S_u + B_u$. One timestep later,

$$c_{uu} = \Delta_u uuS + e^{r\delta t} B_u, \text{and} \tag{1}$$

$$c_{ud} = \Delta_u udS + e^{r\delta t} B_u. \tag{2}$$

Thus, we have two equations in two unknowns. Solving equations (1) and (2) for Δ_u and B_u, we obtain:

$$\Delta_u = \frac{c_{uu} - c_{ud}}{uuS - udS} \geq 0, \text{and} \tag{3}$$

$$B_u = \frac{uc_{ud} - dc_{uu}}{e^{r\delta t}(u-d)} \leq 0. \tag{4}$$

Next, let's consider equations (3) and (4) in light of our numerical example. In order to solve these equations, we must first determine the values for c_{uu} and c_{ud}. Since $K = \$100$, this means that the call option is in-the-money at the uu node (i.e., $c_{uu} = \$10.25$) and out-of-the-money at the ud node (i.e., $c_{ud} = \$0$). Thus, $\Delta_u = \frac{c_{uu} - c_{ud}}{uuS - udS} = \frac{10.25 - 0}{110.25 - 99.75} = .9762$ and $B_u = \frac{uc_{ud} - dc_{uu}}{e^{r\delta t}(u-d)} = \frac{1.05(0) - .95(10.25)}{e^{.05/12}(1.05 - .95)} = -\$96.97$ which implies that $V_{RP}^u = \Delta S_u + B_u = .9762(105) + (-96.97) = 102.50 - 96.97 = \$5.53$. Since the replicating portfolio perfectly mimics the node uu and node ud values for the call option,[2] it follows that the arbitrage-free node u price for the call option is \$5.53.

Similar reasoning provides node d and current prices for the replicating portfolio and the call option. The node d calculation is very simple in this particular numerical example.

[1] These share prices imply that $u = 1.05$ and $d = .95$.

[2] Specifically, note from equation (1) that $c_{uu} = \Delta_u uuS + e^{r\delta t} B_u = .9762(110.25) - e^{.05/12} 96.97 = \$10.25$ and from equation (2) that $c_{ud} = \Delta_u udS + e^{r\delta t} B_u = .9762(99.75) - e^{.05/12} 96.97 = \$0$.

Since the call option is out-of-the-money at nodes ud and dd, it follows that $c_{ud} = c_{dd} = \$0$, which in turn implies that $\Delta_d = B_d = 0$. Thus, $V_{RP}^d = c_d = \$0$.

At the beginning of the binomial tree, the investor forms a portfolio consisting of Δ shares of stock and $\$B$ in riskless bonds; this portfolio is designed to replicate the node u and node d values for the call option (which we know are $c_u = \$5.53$ and $c_d = \$0$ respectively). The cost of this node replicating portfolio is $V_{RP} = \Delta S + B$. One timestep later,

$$c_u = \Delta u S + e^{r\delta t} B, \text{and} \tag{5}$$

$$c_d = \Delta d S + e^{r\delta t} B. \tag{6}$$

Thus, we have two equations in two unknowns. Solving equations (5) and (6) for Δ and B, we obtain:

$$\Delta = \frac{c_u - c_d}{uS - dS} \geq 0, \text{and} \tag{7}$$

$$B = \frac{uc_d - dc_u}{e^{r\delta t}(u-d)} \leq 0. \tag{8}$$

Next, let's consider equations (7) and (8) in light of our numerical example. Since $c_u = \$5.53$ and $c_d = \$0$, $\Delta = \frac{c_u - c_d}{uS - dS} = \frac{5.53 - 0}{105 - 95} = .553$ and $B = \frac{uc_d - dc_u}{e^{r\delta t}(u-d)} = \frac{1.05(0) - .95(5.53)}{e^{.05/12}(1.05 - .95)} = -\$52.32$ which implies that $V_{RP} = \Delta S + B = .553(100) + (-52.32) = 55.30 - 52.32 = \$2.98$. Since the replicating portfolio perfectly mimics the node u and node d values for the call option,[3] it follows that today's arbitrage-free price for the call option is \$2.98.

3 Dynamic Put Option Replication

We continue our analysis by considering (via backward induction) how the replicating portfolio for an otherwise identical put option evolves over time. Suppose the initial price move for the underlying asset is "up". At node u, the investor forms a portfolio consisting of Δ_u shares of stock and $\$B_u$ in riskless bonds; this portfolio is designed to replicate the node uu and node ud values for the put option (given by p_{uu} and p_{ud}). The cost of this node u "replicating" portfolio is $V_{RP}^u = \Delta_u S_u + B_u$. One timestep later,

$$p_{uu} = \Delta_u uuS + e^{r\delta t} B_u, \text{and} \tag{9}$$

$$p_{ud} = \Delta_u udS + e^{r\delta t} B_u. \tag{10}$$

[3]Specifically, note from equation (5) that $c_u = \Delta u S + e^{r\delta t} B = .553(105) - e^{.05/12} 52.53 = \$5.53$ and from equation (6) that $c_d = \Delta d S + e^{r\delta t} B = .553(95) - e^{.05/12} 52.32 = \$0$.

Thus, we have two equations in two unknowns. Solving equations (9) and (10) for Δ_u and B_u, we obtain:

$$\Delta_u = \frac{p_{uu} - p_{ud}}{uuS - udS} \leq 0, \text{and} \tag{11}$$

$$B_u = \frac{up_{ud} - dp_{uu}}{e^{r\delta t}(u-d)} \geq 0. \tag{12}$$

Next, let's consider equations (11) and (12) in light of our numerical example. In order to solve these equations, we must first determine the values for p_{uu} and p_{ud}. Since $K = \$100$, this means that the put option is out-of-the-money at the uu node (i.e., $p_{uu} = \$0$) and in-the-money at the ud node (i.e., $p_{ud} = \$0.25$). Thus, $\Delta_u = \frac{p_{uu} - p_{ud}}{uuS - udS} = \frac{0 - 0.25}{110.25 - 99.75} = -.0238$ and $B_u = \frac{up_{ud} - dp_{uu}}{e^{r\delta t}(u-d)} = \frac{1.05(0.25) - .95(0)}{e^{.05/12}(1.05 - .95)} = \$2.61$ which implies that $V_{RP}^u = \Delta_u S_u + B_u = -.0238(105) + 2.61 = -2.50 + 2.61 = \$0.11$. Since the replicating portfolio perfectly mimics the node uu and node ud values for the put option,[4] it follows that the arbitrage-free node u price for the put option is \$0.11.

Similar reasoning provides node d and current prices for the replicating portfolio and the call option. At node d, the investor forms a portfolio consisting of Δ_d shares of stock and $\$B_d$ in riskless bonds; this portfolio is designed to replicate the node ud and node dd values for the put option (given by p_{ud} and p_{dd}). The cost of this node d "replicating" portfolio is $V_{RP}^d = \Delta_d S_d + B_d$. One timestep later,

$$p_{ud} = \Delta_d udS + e^{r\delta t} B_d, \text{and} \tag{13}$$

$$p_{dd} = \Delta_d ddS + e^{r\delta t} B_d. \tag{14}$$

Thus, we have two equations in two unknowns. Solving equations (13) and (14) for Δ_d and B_d, we obtain:

$$\Delta_d = \frac{p_{ud} - p_{dd}}{udS - ddS} \leq 0, \text{and} \tag{15}$$

$$B_d = \frac{up_{dd} - dp_{ud}}{e^{r\delta t}(u-d)} \geq 0. \tag{16}$$

Next, let's consider equations (15) and (16) in light of our numerical example. In order to solve these equations, we must first determine the values for p_{ud} and p_{dd}. Since $K = \$100$, this means that the $p_{ud} = \$0.25$ and $p_{dd} = \$9.75$. Thus, $\Delta_d = \frac{p_{ud} - p_{dd}}{udS - ddS} = \frac{0.25 - 9.75}{99.75 - 90.25} = -1.0$ and $B_d = \frac{up_{dd} - dp_{ud}}{e^{r\delta t}(u-d)} = \frac{1.05(9.75) - .95(0.25)}{e^{.05/12}(1.05 - .95)} = \$99.58$ which implies that $V_{RP}^d =$

[4] Specifically, note from equation (9) that $p_{uu} = \Delta_u uuS + e^{r\delta t} B_u = -.0238(110.25) + e^{.05/12} 2.61 = \$0$ and from equation (10) that $p_{ud} = \Delta_u udS + e^{r\delta t} B_u = -.0238(99.75) + e^{.05/12} 2.61 = \$0.25$.

$\Delta_d S_d + B_d = -1(95) + 99.58 = \$4.58$. Since the replicating portfolio perfectly mimics the node ud and node dd values for the put option,[5] it follows that the arbitrage-free node d price for the put option is \$4.58.

At the beginning of the binomial tree, the investor forms a portfolio consisting of Δ shares of stock and $\$B$ in riskless bonds; this portfolio is designed to replicate the node u and node d values for the put option (which we know are $p_u = \$0.11$ and $p_d = \$4.58$ respectively). The cost of this replicating portfolio is $V_{RP} = \Delta S + B$. One timestep later,

$$p_u = \Delta uS + e^{r\delta t}B, \text{ and} \tag{17}$$

$$p_d = \Delta dS + e^{r\delta t}B. \tag{18}$$

Thus, we have two equations in two unknowns. Solving equations (17) and (18) for Δ and B, we obtain:

$$\Delta = \frac{p_u - p_d}{uS - dS} \leq 0, \text{ and} \tag{19}$$

$$B = \frac{up_d - dp_u}{e^{r\delta t}(u - d)} \geq 0. \tag{20}$$

Next, let's consider equations (19) and (20) in light of our numerical example. Since $p_u = \$0.11$ and $p_d = \$4.58$, $\Delta = \frac{p_u - p_d}{uS - dS} = \frac{.11 - 4.58}{105 - 95} = -.447$ and $B = \frac{up_d - dp_u}{e^{r\delta t}(u-d)} = \frac{1.05(4.58) - .95(0.11)}{e^{.05/12}(1.05 - .95)} = \$46.85$ which implies that $V_{RP} = \Delta S + B = -.447(100) + 46.85 = \$2.15$. Since the replicating portfolio perfectly mimics the node u and node d values for the put option,[6] it follows that today's arbitrage-free price for the put option is \$2.11.

[5]Specifically, note from equation (13) that $p_{ud} = \Delta_d udS + e^{r\delta t}B_d = -1(99.75) + e^{.05/12}99.58 = \$0.25$ and from equation (14) that $p_{dd} = \Delta_d ddS + e^{r\delta t}B_d = -1(90.25) + e^{.05/12}99.58 = \$9.75$.

[6]Specifically, note from equation (17) that $p_u = \Delta uS + e^{r\delta t}B = -.447(105) + e^{.05/12}46.85 = \$0.11$ and from equation (18) that $p_d = \Delta dS + e^{r\delta t}B = -.447(95) - e^{.05/12}46.85 = \$4.58$.