Normal and Standard Normal Distribution

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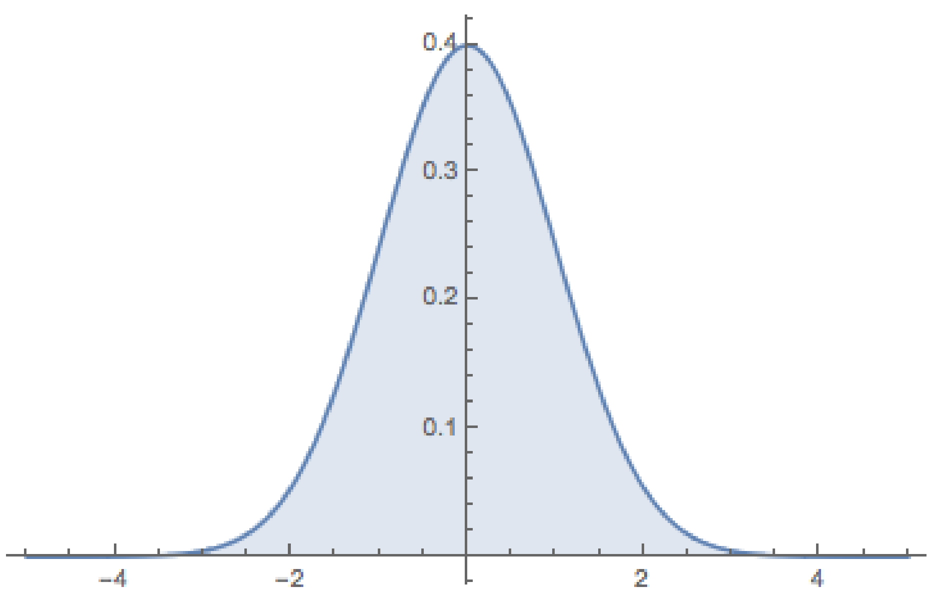
- A continuous random variable x has a *normal distribution* if its probability density function is
$$f(x) = \frac{e^{-.5((x-\mu_x)/\sigma_x)^2}}{\sigma_x\sqrt{2\pi}},$$
where $\sigma_x > 0, -\infty < \mu_x < \infty, -\infty < x < \infty$, and $\int_{-\infty}^{\infty} f(x)dx = 1.0$.

- The normal probability density function has two parameters: μ_x (mean, or expected value) and σ_x (standard deviation).

- Next, we define the *standard normal distribution*. This involves transforming the normal random variable x into a standard normal random variable $z = (x\text{-}\mu_x)/\sigma_x$. Applying the expected value operator to both sides of this equation, $E(z) = (E(x)\text{-}\mu_x)/\sigma_x = 0$ (since $E(x) = \mu_x$). Furthermore, $\sigma_z^2 = E[(z - E(z))^2] = E(z^2)$ (since $E(z) = 0$). Therefore, $\sigma_z^2 = E(z^2) = E\left(\left\{\frac{x-\mu_x}{\sigma_x}\right\}^2\right) = \left(1/\sigma_x^2\right) E\left[(x-\mu_x)^2\right] = \left(\sigma_x^2/\sigma_x^2\right) = 1$, which implies that $\sigma_z = 1$. Thus, the *standard normal density function* is $f(z) = \frac{e^{-z^2/2}}{\sqrt{2\pi}}$.

- Here is a graph of the standard normal density function, $f(z)$:



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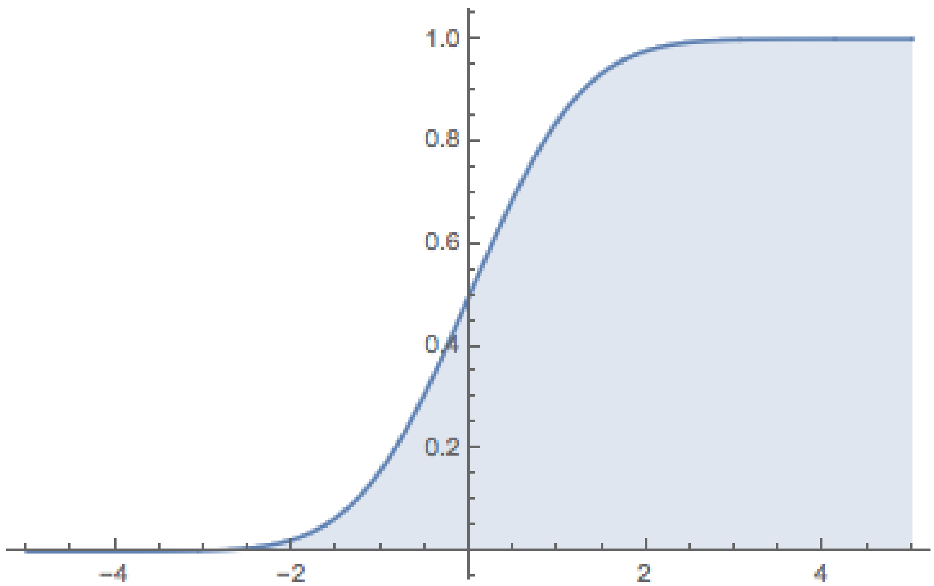
Note that $f(z)$ is symmetric; half of its area is to the left of 0, and the other half is located to the right. Furthermore, although z is unbounded from below and above, nearly 100% of its area occurs within plus and minus 4 standard deviations from its mean (recall that $\sigma_z = 1$).

- Next, we calculate and graph the cumulative distribution function for z, $F(z) = \int_{-\infty}^{z} f(z)dz$:



Note that to the left of z's mean of 0, $F(z)$ is convex (increasing at an increasing rate), whereas to the right of 0, $F(z)$ is concave (increasing at a decreasing rate). As z becomes a "large" positive number, $F(z)$ approaches a value of 1. Also, the slope of $F(z)$ is equal to $f(z)$ for all values of z.

- Calculating $F(z)$: Thankfully, we don't have to literally integrate the standard normal density function in order to calculate probabilities. Excel has a built-in function called NORMSDIST; when you use the command =NORMSDIST(z), Excel returns the probability. For example, if you type = NORMSDIST(0), Excel performs the following calculation: $F(0) = \int_{-\infty}^{0} f(z)dz = 50\%$. If you were interested in knowing what the probability is of being within plus or minus 1 standard deviation from the mean, then by typing =NORMSDIST(1)-NORMSDIST(-1), Excel performs the following calculation: F(1) - F(-1) = $\int_{-1}^{1} f(z)dz$= 68.27%, and so forth. You can also perform these same calculations by referencing a "z" table (e.g., from the home page of the course website, click on "Formula Sheets", then "Standard Normal Distribution Function ("z") Table").