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Options, Futures and Other Derivatives — Name: SOLUTIONS
Problem Set 6

Problem 1.
Consider a European call option on a non-dividend paying stock with an exercise price of \$50 that expires in 3 months. The riskless interest rate (r) is 5%, the underlying asset is worth \$50, and its volatility (σ) is 20%. Assume continuous compounding; i.e., the one timestep discount factor is $e^{-r\delta t}$. Also assume that:

- 1 timestep, $\delta t = 1/12$ (i.e., there will be 3 monthly timesteps until expiration); and
- The "up" move, $u = e^{\sigma\sqrt{\delta t}}$, and the "down" move $d = e^{-\sigma\sqrt{\delta t}}$.

Apply backward induction and risk neutral valuation to determine the arbitrage-free call option prices 1) one month prior to expiration (at nodes uu, ud, and dd), 2) months prior to expiration (at nodes u and d), and 3) three months prior to expiration (at the very beginning of the binomial tree).

SOLUTION: We begin by computing the tree of stock prices over the course of the next 3 months; note that $u = e^{\sigma\sqrt{\delta t}} = e^{.2\sqrt{1/12}} = 1.0594$, and $d = 1/u = 1/1.0594 = .9439$. Thus,

			S_{uuu}= \$59.46
		S_{uu}= \$56.12	
	S_u= \$52.97		S_{uud}= \$52.97
S= \$50		S_{ud}= \$50	
	S_d= \$47.20		S_{udd}= \$47.20
		S_{dd}= \$44.55	
			S_{ddd}= \$42.05

- Since the exercise price for this option is \$50, this implies that $c_{udd} = c_{ddd} = \$0$, whereas $c_{uud} = \$2.97$ and $c_{uuu} = \$9.46$. Next, we calculate values for c_{uu}, c_{ud}, and c_{dd} using the single timestep risk neutral valuation equation:

$$\begin{aligned}
c_{uu} &= e^{-r\delta t}\left[qc_{uuu} + (1-q)c_{uud}\right] = e^{-.05/12}\left[.522(9.46) + .478(2.97)\right] = \$6.33 \\
c_{ud} &= e^{-r\delta t}\left[qc_{uud} + (1-q)c_{udd}\right] = e^{-.05/12}\left[.522(9.46)\right] = \$1.54 \\
c_{ud} &= \$0
\end{aligned}$$

- Next, we calculate values for c_u and c_d using the single timestep risk neutral valuation equation:

$$\begin{aligned} c_u &= e^{-r\delta t}\left[qc_{uu} + (1-q)c_{ud}\right] = e^{-.05/12}\left[.522(6.33) + .478(1.54)\right] = \$4.02 \\ c_d &= e^{-r\delta t}\left[qc_{ud} + (1-q)c_{dd}\right] = e^{-.05/12}\left[.522(1.54)\right] = \$0.80 \end{aligned}$$

- Finally, we calculate the current arbitrage-free price c for this call option applying the single timestep risk neutral valuation equation to c_u and c_d:

$$c = e^{-r\delta t}\left[qc_u + (1-q)c_d\right] = e^{-.05/12}\left[.522(4.02 + .478(0.80)\right] = \$2.47$$

Thus we have the following binomial tree of call option prices:

			c_{uuu}= \$9.46
		c_{uu}= \$6.33	
	c_u= \$4.02		c_{uud}= \$2.97
c = \$2.47		c_{ud}= \$1.54	
	c_d= \$0.80		c_{udd}= \$0
		c_{dd}= \$0	
			c_{ddd}= \$0

Problem 2.
A stock price is currently \$50. It is known that at the end of six months it will be either \$60 or \$42. The risk-free rate of interest with continuous compounding is 12% per year. Calculate the value of a six-month European put option on the stock with an exercise price of \$48 in three different ways: 1) via the delta hedging approach, 2) via the replicating portfolio approach, and 3) via the risk neutral valuation approach.

SOLUTION:

- **Delta Hedging Approach.**
  At the end of six months the value of the option will be either \$0 (if the stock price is \$60) or \$6 (if the stock price is \$42). Consider a portfolio consisting of Δ shares and one put option. The value of the portfolio is either $V_H^u = 60\Delta$ or $V_H^d = 42\Delta + 6$ in six months. Setting $V_H^u = V_H^d$ and solving for Δ, we find that $\Delta = 1/3$, which implies that $V_H^u = V_H^d = 20$.

  The current value of the riskless hedge portfolio is $1/3 \times 50 + p$, where p corresponds to the value of the put option. Since this portfolio must earn the risk-free rate of interest (otherwise there would be a zero net investment, zero risk, positive return arbitrage opportunity), it follows that

  $$1/3 \times 50 + p = 20e^{0.12 \times 0.5},$$

  which implies that $p = \$2.17$.

- **Replicating Portfolio Approach**
  Under the replicating portfolio approach, we form a portfolio consisting of Δ shares and $\$B$ worth of riskless bonds, where $\Delta = \frac{p_u - p_d}{uS - dS} = \frac{0-6}{60-42} = -1/3$ and $B = \frac{up_d - dp_u}{e^{r\delta t}(u-d)} = \frac{1.2(6) - .84(0)}{e^{.12/2}(.36)} = \$18.84$. Thus, the current value of the replicating portfolio is $V_{RP} = \Delta S + B = (-1/3)\$50 + \$18.84 = \$2.17$.

- **Risk Neutral Valuation Approach.**
  This price may also be found by applying risk-neutral valuation. Since the risk neutral probability of an up move is $q = \frac{e^{r\delta t} - d}{u - d} = \frac{1.062 - .84}{1.2 - .84} = .6162$, it follows that $p = e^{-r\delta t}[qp_u + (1-q)p_d] = .9418[.3838(6)] = \$2.17$.