Risk Neutral Valuation Class Problems based on the "Black-Scholes-Merton Model" chapter

Problem 1. Consider a long forward contract that matures at time T and has a delivery price of K. At maturity, the value of this contract is:

$$S_T - K.$$

1. Use risk-neutral valuation to calculate the value of the forward contract at time t in term of the time t price, S_t.

   Applying risk neutral valuation, we can find the date t value (f) of this forward contract by discounting the risk neutral expected value of its maturity value at the riskless rate of interest:

   $$f = e^{-r(T-t)}\hat{E}(S_T - K) = e^{-r(T-t)}\hat{E}(S_T) - Ke^{-r(T-t)},$$

   where $\hat{E}(S_T - K)$ represents the risk neutral expected value of the value of the forward contract at maturity. Since $\hat{E}(S_T) = Se^{r(T-t)}$, date t value (f) of the forward contract may be written as:

   $$f = S - Ke^{-r(T-t)}.$$

2. Confirm that the current value of the forward contract satisfies the Black-Scholes-Merton equation given by $\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf$.

Since we wish to confirm that the value of the forward contract satisfies the Black-Scholes-Merton equation, we must determine $\frac{\partial f}{\partial t}$, $\frac{\partial f}{\partial S}$, and $\frac{\partial^2 f}{\partial S^2}$ for $f = S - Ke^{-r(T-t)}$. Since $\frac{\partial f}{\partial t} = -rKe^{-r(T-t)}$, $\frac{\partial f}{\partial S} = 1$, and $\frac{\partial^2 f}{\partial S^2} = 0$, substituting these comparative static results into the Black-Scholes-Merton equation yields the following result:

$$rf = -rKe^{-r(T-t)} + rS = r(S - Ke^{-r(T-t)})$$

Hence, the Black-Scholes-Merton equation is satisfied.

Problem 2. Assume that a non-dividend-paying stock has an expected return of μ and a volatility of σ. An innovative financial institution has just announced that it will trade a derivative that pays off a dollar amount equal to ln S_T at time T where S_T denotes the values of the stock price at time T.

1. Use risk-neutral valuation to calculate the price of the derivative at time t in term of the time t stock price, S_t.

   On page 13 of the Wiener Processes and Ito's Lemma lecture note, we show that $\ln S_T$ is normally distributed with mean $(\ln S_t + (\mu - .5\sigma^2))(T-t)$ and variance $\sigma^2(T-t)$. Therefore,

$$E(\ln S_T) = \ln S_t + (\mu - \sigma^2/2)(T-t).$$

   We obtain the *risk neutral* expected value of ln S_T by replacing the expected return (μ) with the riskless rate of return (r). The logic behind this parameter substitution is the notion that in a risk-neutral world, investors do not require compensation for bearing risk; they only expect to be compensated for the time value of money. Thus,

$$\hat{E}(\ln S_T) = \ln S_t + (r - \sigma^2/2)(T-t),$$

where the $\hat{E}(\cdot)$ operator indicates a risk neutral expected value. Applying risk-neutral valuation, the value of the derivative at time t is

$$f = e^{-r(T-t)}\hat{E}(\ln S_T) = e^{-r(T-t)}[\ln S + (r - \sigma^2/2)(T-t)].$$

2. Confirm that your price satisfies the Black-Scholes-Merton equation given by $\dfrac{\partial f}{\partial t} + rS\dfrac{\partial f}{\partial S} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 f}{\partial S^2} = rf.$

   Since we wish to confirm that the pricing equation given by $f = e^{-r(T-t)}[\ln S + (r - \sigma^2/2)(T-t)]$ satisfies the Black-Scholes-Merton equation, we need to determine the following set of partial derivatives for f and substitute them back into the BSM equation:

   - $\dfrac{\partial f}{\partial t} = re^{-r(T-t)}[\ln S + (r - \sigma^2/2)(T-t)] - e^{-r(T-t)}\left(r - \sigma^2/2\right),$
   - $\dfrac{\partial f}{\partial S} = \dfrac{e^{-r(T-t)}}{S}$, and
   - $\dfrac{\partial^2 f}{\partial S^2} = -\dfrac{e^{-r(T-t)}}{S^2}.$

   Substituting our expressions for theta $(\dfrac{\partial f}{\partial t})$, delta $(\dfrac{\partial f}{\partial S})$, and gamma $(\dfrac{\partial^2 f}{\partial S^2})$

into the left-hand side of the Black-Scholes-Merton equation $(\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf)$, we obtain:

$$
\begin{aligned}
rf &= \frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} \\
&= re^{-r(T-t)}[\ln S + (r - \sigma^2/2)(T-t)] - e^{-r(T-t)}\left(r - \sigma^2/2\right) + rS\frac{e^{-r(T-t)}}{S} - \frac{1}{2}\sigma^2 S^2 \frac{e^{-r(T-t)}}{S^2} \\
&= re^{-r(T-t)}[\ln S + (r - \sigma^2/2)(T-t)] - e^{-r(T-t)}\left(r - \sigma^2/2\right) + e^{-r(T-t)}\left(r - \sigma^2/2\right) \\
&= re^{-r(T-t)}[\ln S + (r - \sigma^2/2)(T-t)].
\end{aligned}
$$

Problem 3. A financial institution plans to offer a security that pays off a dollar amount equal to S_T^2 at time T.

1. Use risk-neutral valuation to calculate the price of the security at time t in terms of the stock price, S, at time t. (Hint: The expected value of S_T^2, $E(S_T^2)$, can be inferred from the equations given for the mean and variance of a lognormal distribution which appear in the Black-Scholes-Merton Model chapter; $E(S_T) = S_t e^{\mu(T-t)}$, and $Var\,(S_T) = S_t^2 e^{2\mu(T-t)}(e^{\sigma^2(T-t)} - 1)$.

   The expected value of the payoff on this security is $E(S_T^2)$. In order to calculate $E(S_T^2)$, we will use the above referenced equations for $E(S_T)$ and $Var\,(S_T)$. The general definition for $Var\,(S_T)$ is:

$$\begin{aligned} Var\,(S_T) &= E[(S_T - E(S_T))^2] \\ &= E[S_T^2 + E(S_T)^2 - 2S_T E(S_T)] \\ &= E(S_T^2) + E(S_T)^2 - 2E(S_T)^2 \\ &= E(S_T^2) - E(S_T)^2. \end{aligned}$$

   Solving for $E(S_T^2)$, we obtain $E(S_T^2) = Var\,(S_T) + E(S_T)^2$. Therefore,

$$E(S_T^2) = S_t^2 e^{2\mu(T-t)}(e^{\sigma^2(T-t)} - 1) + S_t^2 e^{2\mu(T-t)} = S_t^2 e^{(2\mu+\sigma^2)(T-t)}.$$

In a risk-neutral world $\mu = r$, so the risk neutral expected value of $\hat{E}(S_T^2) = S^2 e^{(2r+\sigma^2)(T-t)}$. Thus, by risk-neutral valuation, the value of the derivative security at time t is

$$f = e^{-r(T-t)}\hat{E}(S_T^2) = S^2 e^{(2r+\sigma^2)(T-t)} e^{-r(T-t)} = S^2 e^{(r+\sigma^2)(T-t)}.$$

2. Confirm that your price satisfies the Black-Scholes-Merton equation given by $\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf.$

Since we wish to confirm that the pricing equation given by $f = S^2 e^{(r+\sigma^2)(T-t)}$ satisfies the Black-Scholes-Merton equation, we must calculate the following set of partial derivatives for f:

$$\frac{\partial f}{\partial t} = -S^2(r+\sigma^2)e^{(r+\sigma^2)(T-t)},$$

$$\frac{\partial f}{\partial S} = 2Se^{(r+\sigma^2)(T-t)}, \text{ and}$$

$$\frac{\partial^2 f}{\partial S^2} = 2e^{(r+\sigma^2)(T-t)}.$$

Substituting our expressions for theta $(\frac{\partial f}{\partial t})$, delta $(\frac{\partial f}{\partial S})$, and gamma $(\frac{\partial^2 f}{\partial S^2})$

into the left-hand side of the Black-Scholes-Merton equation ($\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf$), we obtain:

$$
\begin{aligned}
-S^2(r+\sigma^2)e^{(r+\sigma^2)(T-t)} + 2rS^2e^{(r+\sigma^2)(T-t)} + \sigma^2 S^2 e^{(r+\sigma^2)(T-t)} \\
= rS^2e^{(r+\sigma^2)(T-t)} = rf
\end{aligned}
$$

Hence, the Black-Scholes-Merton equation is satisfied.

Problem 4. Suppose that a European call option on a non-dividend paying stock pays off a dollar amount equal to $Max[0, S_T - K]$ at time T.

1. Use risk-neutral valuation to calculate the price of the security at time t in terms of the stock price, S, at time t. (Hint: See the (optional) reading entitled "Derivation and Comparative Statics of the Black-Scholes Call and Put Option Pricing Equations", where this is shown.)

   The price of this call option is:

   $$f = SN(d_1) - Ke^{-r(T-t)}N(d_2),$$

   where $d_1 = \dfrac{\ln(S/K) + (r + .5\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$, $d_2 = d_1 - \sigma\sqrt{(T-t)}$, and $\sigma^2 =$ variance of underlying asset's rate of return, and $N(z) =$ standard normal distribution function evaluated at z.

   The price of the call option depends upon five parameters; specifically, the current (i.e., date t) stock price S, the exercise price K, the riskless interest rate r, the time to expiration $(T - t)$, and the volatility of the underlying asset σ.

2. Confirm that your price satisfies the Black-Scholes-Merton equation given by $\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf$.

As I show in my "Derivation and Comparative Statics of the Black-Scholes Call and Put Option Pricing Equations" teaching note, the call option's "theta" corresponds to the rate at which the value of the call option decays with the passage of time; i.e., $\frac{\partial f}{\partial t} = -rKe^{-r(T-t)}N(d_2) - Sn(d_1)\frac{.5\sigma}{\sqrt{T-t}} < 0$. Furthermore, the call option's delta $\frac{\partial f}{\partial S} = N(d_1) > 0$, and it's gamma is $\frac{\partial^2 f}{\partial S^2} = \frac{\partial N(d_1)}{\partial S} = \frac{\partial N(d_1)}{\partial d_1}\frac{\partial d_1}{\partial S} = n(d_1)\frac{\partial d_1}{\partial S} = \frac{n(d_1)}{S\sigma\sqrt{T-t}} > 0$.

Substituting our expressions for theta $(\frac{\partial f}{\partial t})$, delta $(\frac{\partial f}{\partial S})$, and gamma $(\frac{\partial^2 f}{\partial S^2})$ into the left-hand side of the Black-Scholes-Merton equation $(\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf)$, we obtain:

$$\underbrace{-rKe^{-r(T-t)}N(d_2) - Sn(d_1)\frac{.5\sigma}{\sqrt{(T-t)}}}_{\frac{\partial f}{\partial t}} + \underbrace{rSN(d_1)}_{rS\frac{\partial f}{\partial S}} + \underbrace{.5\sigma^2 S^2 \frac{n(d_1)}{S\sigma\sqrt{(T-t)}}}_{.5\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}} =$$

$$\underbrace{r[SN(d_1) - Ke^{-r(T-t)}N(d_2)]}_{rf}$$

Simplifying and rearranging terms, we confirm that the Black-Scholes-Merton (Black-Scholes-Merton) call option pricing formula satisfies the Black-Scholes-Merton equation:

$$r[SN(d_1) - Ke^{-r(T-t)}N(d_2)] - Sn(d_1)\frac{.5\sigma}{\sqrt{(T-t)}} + Sn(d_1)\frac{.5\sigma}{\sqrt{(T-t)}} =$$
$$r[SN(d_1) - Ke^{-r(T-t)}N(d_2)].$$