Risk Neutral Valuation in Continuous Time Synopsis

1. In the "Geometric Brownian Motion, Itô's Lemma, and Risk Neutral Valuation" lecture note, we learned that if the price of a risky asset evolves over time according to the Geometric Brownian motion equation (given by $dS = \mu S dt + \sigma S dz$), then there exists a *risk neutral valuation relationship* (RNVR) between the price of a derivative based on the price of this risky asset described by the following differential equation (AKA the "Black-Scholes-Merton" equation):

$$rf = \frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}. \tag{1}$$

2. Equation (1) consists of three key components: 1) a *theta* term, which captures how the price of a derivative changes over time, 2) a *delta* term, which captures the sensitivity of a derivative's price to changes in the price of the underlying asset, and 3) a *gamma* term which captures how changes in delta are influenced by the variability of the price of an underlying asset.

3. The reason why the Black-Scholes-Merton equation is so important is because it provides the framework needed for determining arbitrage-free prices of virtually any derivative security which references S as its underlying asset price. For example, Black and Scholes derived their famous option pricing formula

by solving a (slightly) transformed version of this equation subject to the "boundary" condition that the payoff of a European call option at time T is $C_T = Max[S_T - K, 0]$, where K represents the option's exercise price and S_T represents the underlying asset price at expiration.

4. Another important implication of the Black-Scholes-Merton equation is that the price of a derivative is equal to the present value of the *risk neutral expected value* of its payoff(s). For derivatives which do not have bounded payoffs such as options, this is a particularly simple calculation. All one must do is 1) calculate the expected value of the payoff, 2) transform it into a risk neutral expected value by substituting the riskless interest rate r in place of the expected return μ, and 3) discount the risk neutral expected value back to the present using the riskless rate of interest. If done correctly, such a pricing formula will satisfy the Black-Scholes-Merton equation.