Risk Neutral versus "True" Probability of Default (and Comparative Statics)

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In our application of the Black-Scholes-Merton option pricing model to evaluating credit risk, we show that the value of the "option to default" corresponds to a put option on the value of the firm's assets ($V(F)$) with an exercise price which correspond to the promised payment on a zero coupon bond (B).

The equation for this put option is as follows:

$$V(\max(0, B - F)) = Be^{-rT}N(-d_2) - V(F)N(-d_1),$$

where $d_1 = \frac{\ln(V(F)/B) + (r + .5\sigma^2)T}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$. In the above equation, $N(-d_2)$ corresponds to the risk neutral probability of default; i.e., it corresponds to the risk neutral probability that $B > F$ at date T.

Of greater interest for risk management purposes is how to determine the "true" probability that default will occur. We obtain an expression for the appropriate z statistic for this problem by reverse engineering our expression for $-d_2$; i.e., by transitioning back from risk neutrality to risk aversion. To see this, we begin by solving for d_2 in terms of d_1:

$$\begin{aligned} d_2 &= d_1 - \sigma\sqrt{T}; \\ &= \frac{\ln(V(F)/B) + (r + .5\sigma^2)T}{\sigma\sqrt{T}} - \frac{\sigma^2 T}{\sigma\sqrt{T}}; \\ &= \frac{\ln(V(F)/B) + (r - .5\sigma^2)T}{\sigma\sqrt{T}}. \end{aligned}$$

Thus, $-d_2 = -\frac{\ln(V(F)/B) + (r - .5\sigma^2)T}{\sigma\sqrt{T}} = \frac{\ln(B/V(F)) - (r - .5\sigma^2)T}{\sigma\sqrt{T}}$.

To complete the transition from risk neutrality to risk aversion, we replace the riskless rate of return r with the expected rate of return μ; we'll refer to this z statistic as "$z_{default}$":

$$z_{default} = \frac{\ln(B/V(F)) - (\mu - .5\sigma^2)T}{\sigma\sqrt{T}}.$$

Therefore, the "true" probability of default is simply

$$N(z_{default}) = N\left(\frac{\ln(B/V(F)) - (\mu - .5\sigma^2)T}{\sigma\sqrt{T}}\right).$$

Next, consider the comparative statics of the probability of default; since higher values for the $z_{default}$ variable imply higher probabilities of default, we'll sign the following set of partial derivatives: $\frac{\partial z_{default}}{\partial B}$, $\frac{\partial z_{default}}{\partial V(F)}$, $\frac{\partial z_{default}}{\partial \mu}$, $\frac{\partial z_{default}}{\partial \sigma}$, and $\frac{\partial z_{default}}{\partial T}$:

$$\begin{aligned}
\frac{\partial z_{default}}{\partial B} &= \frac{1}{B\sigma\sqrt{T}} > 0; \\
\frac{\partial z_{default}}{\partial V(F)} &= -\frac{1}{V(F)\sigma\sqrt{T}} < 0; \\
\frac{\partial z_{default}}{\partial \mu} &= -\frac{\sqrt{T}}{\sigma} < 0; \\
\frac{\partial z_{default}}{\partial \sigma} &= \frac{\ln(V(F)/B) + (\mu + 0.5\sigma^2)T}{\sigma^2\sqrt{T}} > 0; \text{and} \\
\frac{\partial z_{default}}{\partial T} &= \frac{\ln\left(V(F)/B\right) - (\mu - .5\sigma^2)T}{2\sigma T^{3/2}} <> 0.
\end{aligned}$$

These comparative statics indicate that the probability of default is positively related to the extent to which the firm uses debt financing and how risky corporate assets are. On the other hand, the probability of default is negatively related to the value of and expected return on the firm's assets.

The relationship between the maturity date on bonds and the probability of default is ambiguous; there could be either a positive or negative relationship between probability of default and time to expiration. For example, $\ln(V(F)/B)$ is negative if $B > V(F)$, positive if $B < V(F)$. If $\mu - .5\sigma^2 = 0$, then the negative versus positive sign depends only on whether $\ln(V(F)/B)$ is negative or positive. Generally, we expect $\mu - .5\sigma^2 > 0$, and since $\mu - .5\sigma^2$ is subtracted, then the sign of this derivative depends on how $\ln(V(F)/B)$ and $\mu - .5\sigma^2$ trade off against each other.

We will also consider these comparative statics numerically in class when we discuss the credit risk topic.