A Simple Model of a Financial Market

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1 Introduction

We consider a simple model of a financial market: there are two assets, a bond and a stock. The bond is riskless, hence — by definition — we know what its price will be tomorrow (at time $t = 1$). For simplicity, we assume that tomorrow's bond price is $B_1 = \$1$ and that the rate of interest $r = 0\%$; thus the price of the bond today (at time $t = 0$) is:

$$B_0 = B_1/(1 + r) = B_1 = \$1. \tag{1}$$

The more interesting feature of the model is that the stock is risky; we know its value today, say $S_0 = \$1$, but we do not know (for sure, anyway) its value tomorrow. We model this uncertainty by defining tomorrow's *state-contingent* price as $S_{1,s}$, where s indicates whether the state of tomorrow's economy will be good (in which case $s = g$) or bad (in which case $s = b$). Both states are equally likely; in the good economy state, $S_{1,g} = \$2$, whereas in the bad economy state, $S_{1,b} = \$0.50$. Thus, the *expected value* of S_1, $E(S_1) = .5(\$2) + .5(\$0.50) = \$1.25$. Therefore, the *expected return* on one share of stock is 25%; the stock has a higher expected return than the bond, but it is also risky, since it is equally likely that the stock price will either double or drop by 50 percent!

Next we introduce a third financial instrument: a *call option* on the stock with an exercise, or "strike" price of $\$K$; the buyer of the call option has the right — but not the obligation — to buy one stock tomorrow at the predefined price $\$K$. To fix ideas let $K = 1$. A moment's

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reflection reveals that tomorrow's state-contingent call option price, $C_{1,s}$, is:

$$C_{1,s} = max(S_{1,s} - K, 0). \tag{2}$$

Since $S_{1,g} = \$2$ and $S_{1,b} = \$0.50$, this implies that $C_{1,g} = max(\$2 - \$1, 0) = \$1$ and $C_{1,b} = max(\$0.50 - \$1, 0) = \$0$. Thus, we know tomorrow's call option value, contingent upon tomorrow's stock price. But what is the price of the call option *today*?

2 Classical Approach to Pricing Call Options

The classical approach, used by actuaries for centuries, is to price call options by calculating the present value of the call option's expected payoff, which leads to the value $C_0 = E(C_1)/(1 + r) = E(C_1) = .5(\$1) + .5(\$0) = \$0.50$. Indeed, the rationale behind pricing the call option in this fashion can be found in an important statistical principle called the *law of large numbers*; in the long run, the call option buyer will (on average) neither gain nor lose. The call option buyer is equally likely to either earn profit $\pi_{1,g} = C_{1,g} - C_0 = \$1 - \$0.50 = \$0.50$ or suffer loss $\pi_{1,b} = C_{1,b} - C_0 = \$0.00 - \$0.50 = -\$0.50$, so the expected value of profit $E(\pi) = .5(-\$0.50) + .5(\$0.50) = \$0.00$. Similarly, there is no profit from investing in the riskless bond, so on average, the performance of an investment in the risky call option is equal to the performance of the riskless bond.

However, if investors are *risk averse*, an investment in a risky asset should, on average, yield a better performance than an investment in a safe asset, other things equal. This is certainly the case for our stock and bond; although today's price for both securities is the same — i.e., $S_0 = B_0 = \$1$, the risky stock is expected to perform better than the safe bond, since $E(S_1) = .5(\$2) + .5(\$.50) = \$1.25 > \$1 = B_1$. Assuming that investors are risk averse, they cannot be indifferent about earning the same average profit from investing in a risky call option as they can earn from investing in a riskless bond.[1] Thus risk averse

[1]If investors were indifferent about such an outcome, this would imply that they are *risk neutral*.

investors *would not* be willing to pay a price of $C_0 = \$0.50$ for the call option; they would instead expect to pay a *lower* price so that they can earn a *higher* expected profit from their investment in the risky call option.

3 Arbitrage-Free Approach to Pricing Call Options

Since we assume that investors are risk averse, a different approach for option pricing is called for. Assuming that the stock and bond are correctly valued, we can infer the call option price by determining the value of a "replicating" portfolio consisting of the stock and bond which mimics the state-contingent payoffs on the call option.

Let Δ correspond to the number of shares of stock in the replicating portfolio, and β correspond to the number of bonds. Since $C_{1,g} = \$1$ and $C_{1,b} = \$0$, it follows that the value of the replicating portfolio in the good economy state, $V_{1,g} = \Delta(S_{1,g}) + \beta(B_1) = \Delta(\$2) + \beta(\$1) = \$1$, and the payoff on the replicating portfolio in the bad economy state, $V_{1,b} = \Delta(S_{1,b}) + \beta(B_1) = \Delta(\$.5) + \beta(\$1) = 0$. Thus we have two equations in two unknowns:

$$V_{1,g} = \Delta(\$2) + \beta(\$1) = \$1, \text{and}$$
$$V_{1,b} = \Delta(\$0.50) + \beta(\$1) = \$0.$$

Subtracting the equation for $V_{1,b}$ from the equation for $V_{1,g}$ and solving for Δ, we obtain $\Delta = 2/3$. Substituting $\Delta = 2/3$ back into either the $V_{1,g}$ or $V_{1,b}$ equation, we find that $\beta = -1/3$.

The reader might be puzzled about how to interpret $\beta = -1/3$. Investing a negative amount of money in a bond — "going short" — means to borrow money. Note that if the state of tomorrow's economy is good, then $V_{1,g}$ = (2/3)(S_1) - (1/3)(B_1) = (2/3)(\$2) - (1/3)(\$1) = \$1, and if the state of tomorrow's economy is bad, then $V_{1,b}$ = (2/3)(S_1) - (1/3)(B_1) = (2/3)(\$0.50) - (1/3)(\$1) = \$0. In other words, the long-short stock-bond portfolio "replicates" the call option in both the good and bad states of the economy, i.e.,

$$V_{1,s} = C_{1,s} \text{ for all } s. \quad (3)$$

The replicating portfolio has a well-defined price today; namely, $V_0 = (2/3)(S_0)$ - $(1/3)(B_0)$ = (2/3)(\$1) - (1/3)(\$1) = \$0.33. Next comes the "arbitrage-free" pricing argument: since equation 3 shows that tomorrow's value for the replicating portfolio is equal to tomorrow's value for the call option in all states, it follows that *today's* value for the replicating portfolio must also be equal to today's value for the call option; i.e.,

$$V_0 = C_0 = \$0.33. \quad (4)$$

To see *why* equation 4 *must* obtain, consider a proof by contradiction. Specifically, suppose C_0= \$0.50 as indicated by the classical approach to option pricing discussed earlier. If this were the case, investors could earn positive profits without bearing any risk or having to invest any of their own money by simply buying the replicating portfolio today and funding this purchase by simultaneously selling the call option. Today, the investor would enjoy a cash inflow totaling $C_0 - V_0$ = \$0.50 - \$0.33 = \$0.17, whereas tomorrow the long position in the replicating portfolio and the short position in the call option would completely offset each other, irrespective of whether the state of the economy is good or bad. However, investors would realize that this "free lunch" exists and react by bidding down the price of the call option until its market value becomes C_0 = \$0.33.

4 Arbitrage-Free Approaches to Pricing Put Options

Next we introduce a fourth financial instrument: a *put option* on the stock with an exercise, or "strike" price of $\$K$; the buyer of the put option has the right — but not the obligation — to *sell* one stock tomorrow at the predefined price $\$K$. A moment's reflection reveals that tomorrow's state-contingent call option price, $P_{1,s}$, is:

$$P_{1,s} = max(K - S_{1,s}, 0). \tag{5}$$

Since we already know the value of an otherwise identical [2] call option, we can determine the value of the put option by invoking the put-call parity theorem. In what follows, we will show how to price the put option by applying both the put-call parity and replicating portfolio methods.

4.1 Pricing the Put Option via put-call parity

The put-call parity theorem states that the value of a portfolio consisting of a European call option on a non-dividend paying stock with exercise price K plus the present value of a pure discount bond with a par value of K must be equal to the value of a portfolio consisting of an otherwise identical put option plus one share.[3] Thus,

$$C_0 + K/(1+r) = P_0 + S_0 \Rightarrow P_0 = C_0 + K/(1+r) - S_0 \Rightarrow P_0 = \$.33 + 1 - 1 = \$.33. \tag{6}$$

4.2 Pricing the Put Option's Replicating Portfolio

Alternatively, we may infer the put option price by determining the value of a "replicating" portfolio consisting of the stock and bond which mimics the state-contingent payoffs on the put option.

Our approach to identifying the replicating portfolio for the put option is conceptually similar to the approach we followed for the call option. Let Δ correspond to the number of shares of stock in the replicating portfolio, and β correspond to the number of bonds.

[2] By "otherwise identical", we mean that the put and the call are both written on the same stock and have the same exercise price.

[3] This must be true since both portfolios pay off $Max(S_1, K)$ one period from today. If this weren't true, then the investor could make riskless profits with zero net investment by simply selling the more expensive portfolio and buying the cheaper one.

Since $S_{1,g} = \$2$ and $S_{1,b} = \$0.50$, this implies that $P_{1,g} = max(\$1 - \$2, 0) = \$0$ and $P_{1,b} = max(\$1 - \$0.50, 0) = \$0.50$. In the good economy state, the value of the replicating portfolio $V_{1,g} = \Delta(S_{1,g}) + \beta(B_1) = \Delta(\$2) + \beta(\$1) = \$0$, and in the bad economy state, the value of the replicating portfolio, $V_{1,b} = \Delta(S_{1,b}) + \beta(\$1) = \Delta(\$0.50) + \beta(\$1) = \$0.50$. Thus we have two equations in two unknowns:

$$V_{1,g} = \Delta(\$2) + \beta(\$1) = \$0, \text{and}$$
$$V_{1,b} = \Delta(\$0.50) + \beta(\$1) = \$0.50.$$

Subtracting the equation for $V_{1,b}$ from the equation for $V_{1,g}$ and solving for Δ, we obtain $\Delta = -1/3$. Substituting $\Delta = -1/3$ back into either the $V_{1,g}$ or $V_{1,b}$ equation, we find that $\beta = 2/3$. Note that this portfolio replicates the payoffs on the put option; i.e., if the state of tomorrow's economy is good, then $V_{1,g}$ = (-1/3)(S_1) + (2/3)(B_1) = (-1/3)(\$2) + (2/3)(\$1) = \$0, and if the state of tomorrow's economy is bad, then $V_{1,b}$ = (-1/3)(S_1) + (2/3)(B_1) = (-1/3)(\$0.50) + (2/3)(\$1) = \$0.50. In other words, the short-long stock-bond portfolio "replicates" the put option in both the good and bad states of the economy, i.e.,

$$V_{1,s} = P_{1,s} \text{ for all } s. \tag{7}$$

Since equation 7 indicates that tomorrow's value for the replicating portfolio is equal to tomorrow's value for the put option in all states, it follows that *today's* value for the put option must also be equal to today's value for the replicating portfolio ; i.e.,

$$P_0 = V_0 = -1/3(S_0) + 2/3(B_0) = -1/3(\$1) + 2/3(\$1) = \$0.33. \tag{8}$$

5 Arbitrage-Free Forward Contract Pricing

Finally, we introduce a fifth financial instrument: a *forward contract* on the stock with a *delivery*, or *forward* price denoted by $\$K$. If we *buy* forward today, the payoff on this long forward contract tomorrow is $f_1^l = S_1 - K$; however, if we *sell* forward today, the payoff on this short forward contract tomorrow is $f_1^s = K - S_1$. Since

$$f_1^l = S_{1,s} - K = Max(S_{1,s} - K, 0) - Max(K - S_{1,s}, 0), \text{ and} \tag{9}$$

$$f_1^s = K - S_{1,s} = Max(K - S_{1,s}, 0) - Max(S_{1,s} - K, 0), \tag{10}$$

it follows that the replicating portfolio for a long (short) forward contract may be created by buying (selling) a call and selling (buying) a put on the same underlying asset, where the exercise price for these options is set equal to the initial forward price. This in turn implies that the initial value for both the long and short forward contracts is equal to \$0, since the call and put prices in this example perfectly offset each other.