Black-Scholes-Merton Call and Put Equations and Comparative Statics

1 Black-Scholes-Merton Option Pricing Summary

- The Black-Scholes-Merton formula for the value of a European call option is: $c = SN(d_1) - Ke^{-r\tau}N(d_2)$, where $d_1 = \dfrac{\ln(S/K) + \left(r + .5\sigma^2\right)\tau}{\sigma\sqrt{\tau}}$, $d_2 = d_1 - \sigma\sqrt{\tau}$, and $\tau = T - t$; i.e., τ represents the time to expiration for the option.

- The Black-Scholes-Merton formula for the value of a European put option is $p = Ke^{-r\tau}N(-d_2) - SN(-d_1)$.

- The comparative statics of call and put option prices with respect to S, K, r, t, and σ are as follows:

  - **Asset Price ("delta"):** $\partial c/\partial S = N(d_1) > 0$ and $\partial p/\partial S = -N(-d_1) < 0$.

  - **Exercise Price:** $\partial c/\partial K = -e^{-r\tau}N(d_2) < 0$ and $\partial p/\partial K = e^{-r\tau}N(-d_2) > 0$.

  - **Interest Rate ("rho")**: $\partial c/\partial r = \tau Ke^{-r\tau}N(d_2) > 0$ and $\partial p/\partial r = -\tau Ke^{-r\tau}N(-d_2) < 0$.

- **Time to Expiration ("theta"):** $\partial c/\partial t = -Sn(d_1)\frac{.5\sigma}{\sqrt{\tau}} - rKe^{-r\tau}N(d_2) < 0$ and $\partial p/\partial t = -Sn(d_1)\frac{.5\sigma}{\sqrt{\tau}} + rKe^{-r\tau}N(-d_2) <> 0$.

- **Volatility ("vega"):** $\partial c/\partial \sigma = Sn(d_1)\sqrt{\tau}$ and $\partial p/\partial \sigma = Sn(-d_1)\sqrt{\tau}$. Thus, $\partial c/\partial \sigma = \partial p/\partial \sigma > 0$.

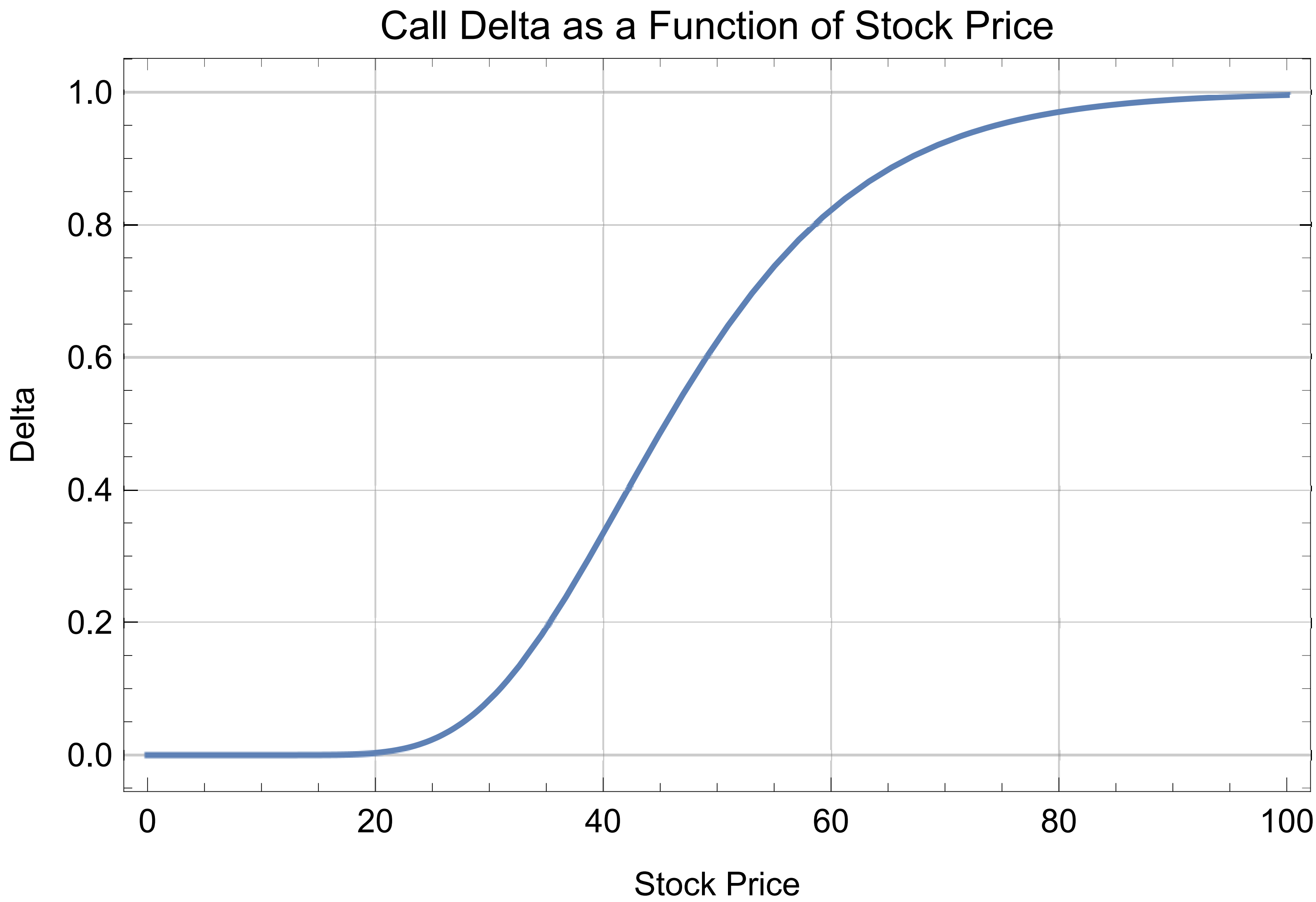
- **Gamma:** $\partial^2 c/\partial S^2 = \partial^2 p/\partial S^2 = n(d_1)/\, S\sigma\sqrt{\tau} > 0$.

2 Numerical Simulations

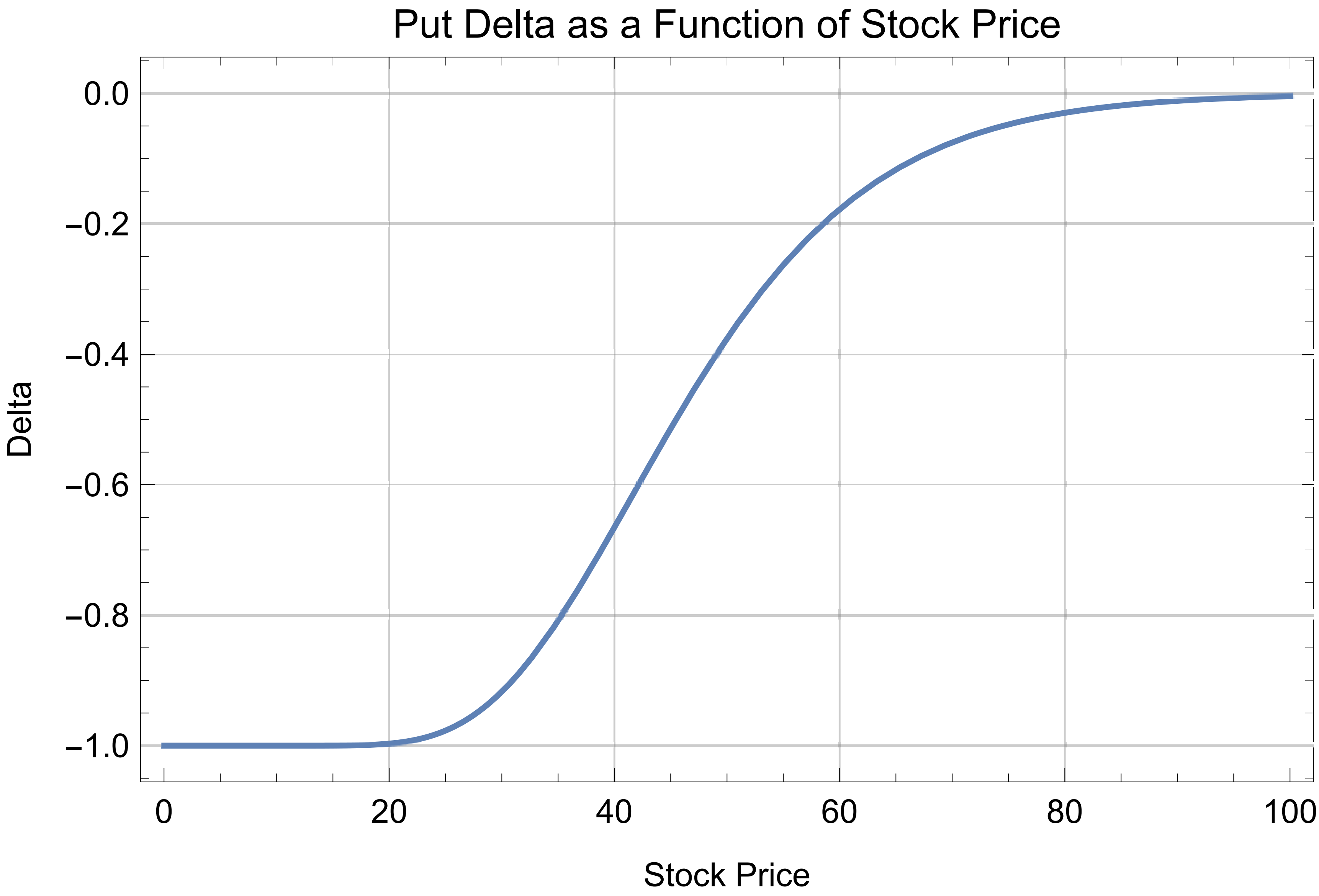
The base case for this numerical simulation involves the following set of parameters: 1) the current (date t) stock price is S = \$50, 2) the exercise price is K = \$50, 3) the (continuously compounded) annualized rate of interest is $r = 5\%$, 4) the time to expiration $\tau = T\text{-}t = 1$ year, and 5) annualized volatility is $\sigma = 30\%$. The Black-Scholes-Merton price of a call option using these parameters is c = \$7.12, and the corresponding put price is p = \$4.68.

2.1 Relationship between Option Deltas and price of underlying

First, we test the relationship between the call option delta and the price of the underlying:

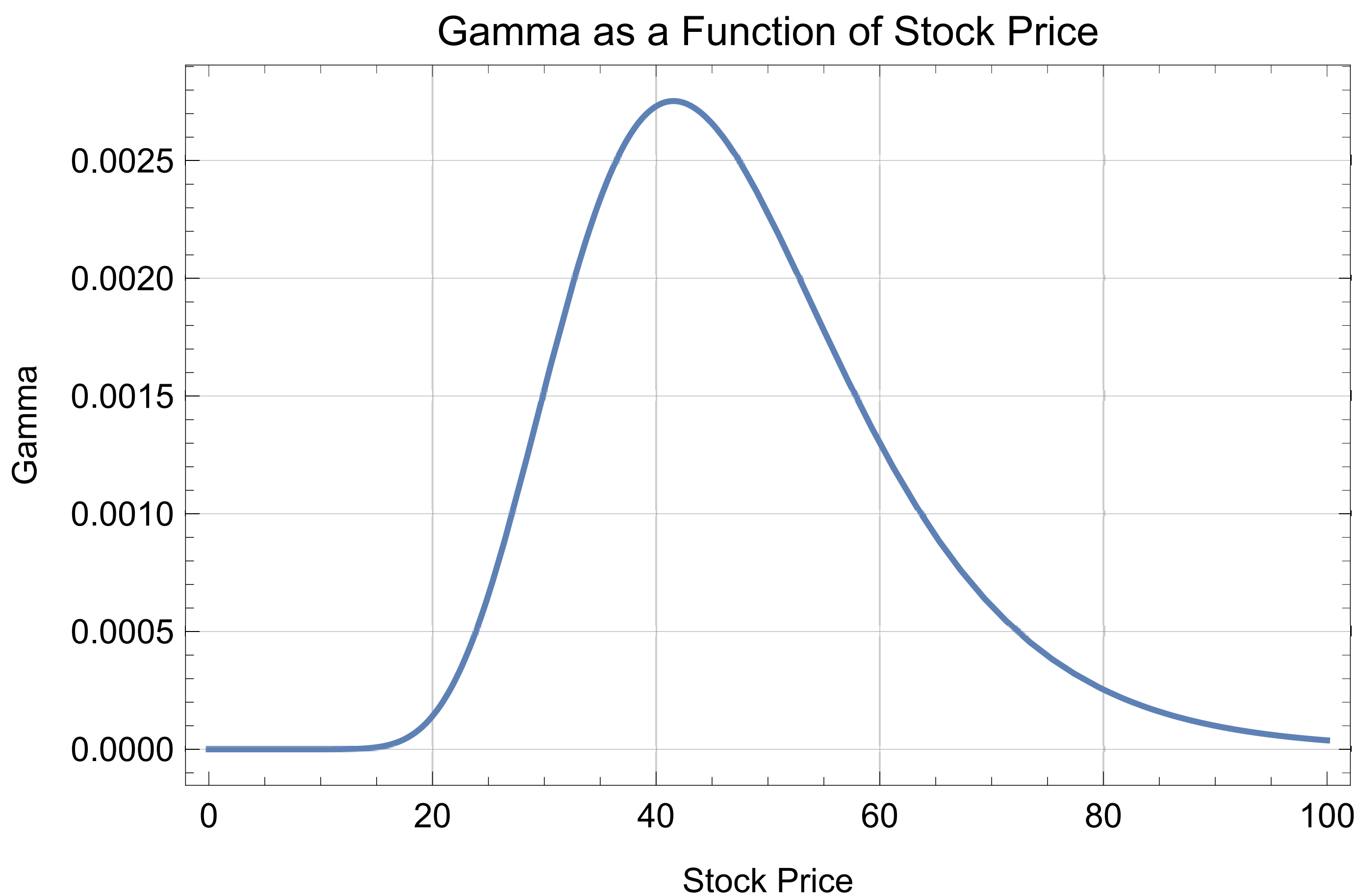


Next, we test the relationship between the put hedge ratio and the price of the underlying:

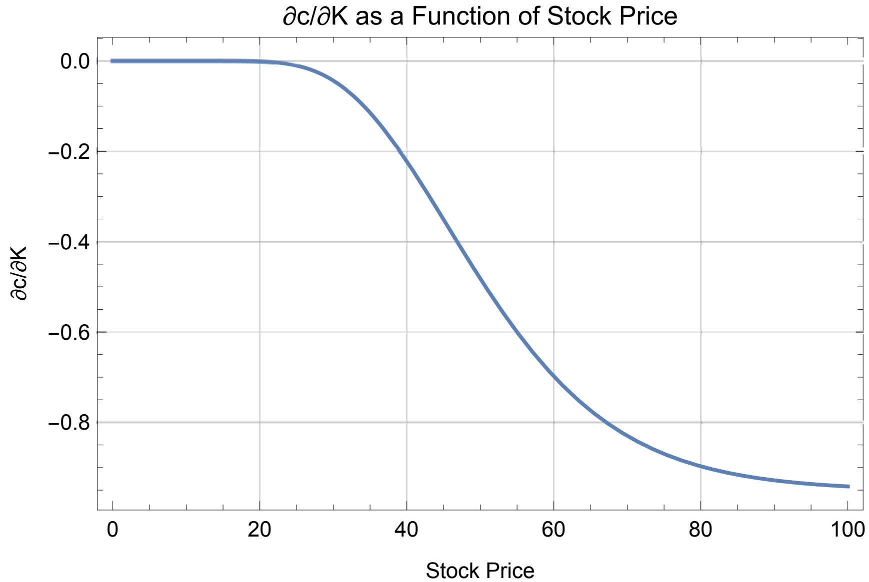


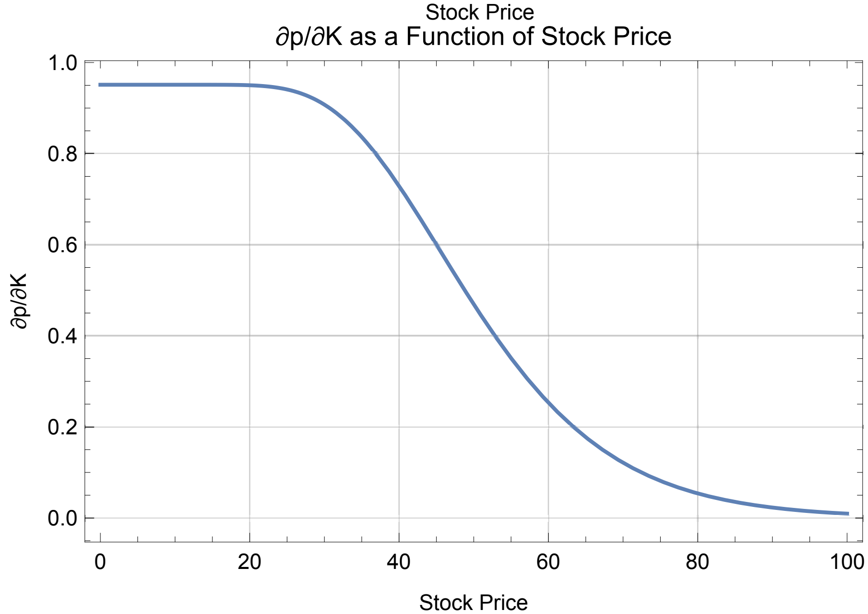
2.2 Relationship between Option Gammas and the price of the underlying

The call option's gamma is identical to the put option gamma; specifically, $\dfrac{\partial^2 c}{\partial S^2} = \dfrac{\partial^2 p}{\partial S^2} = n(d_1)/S\sigma\sqrt{\tau}$. Here's the graph for gamma:

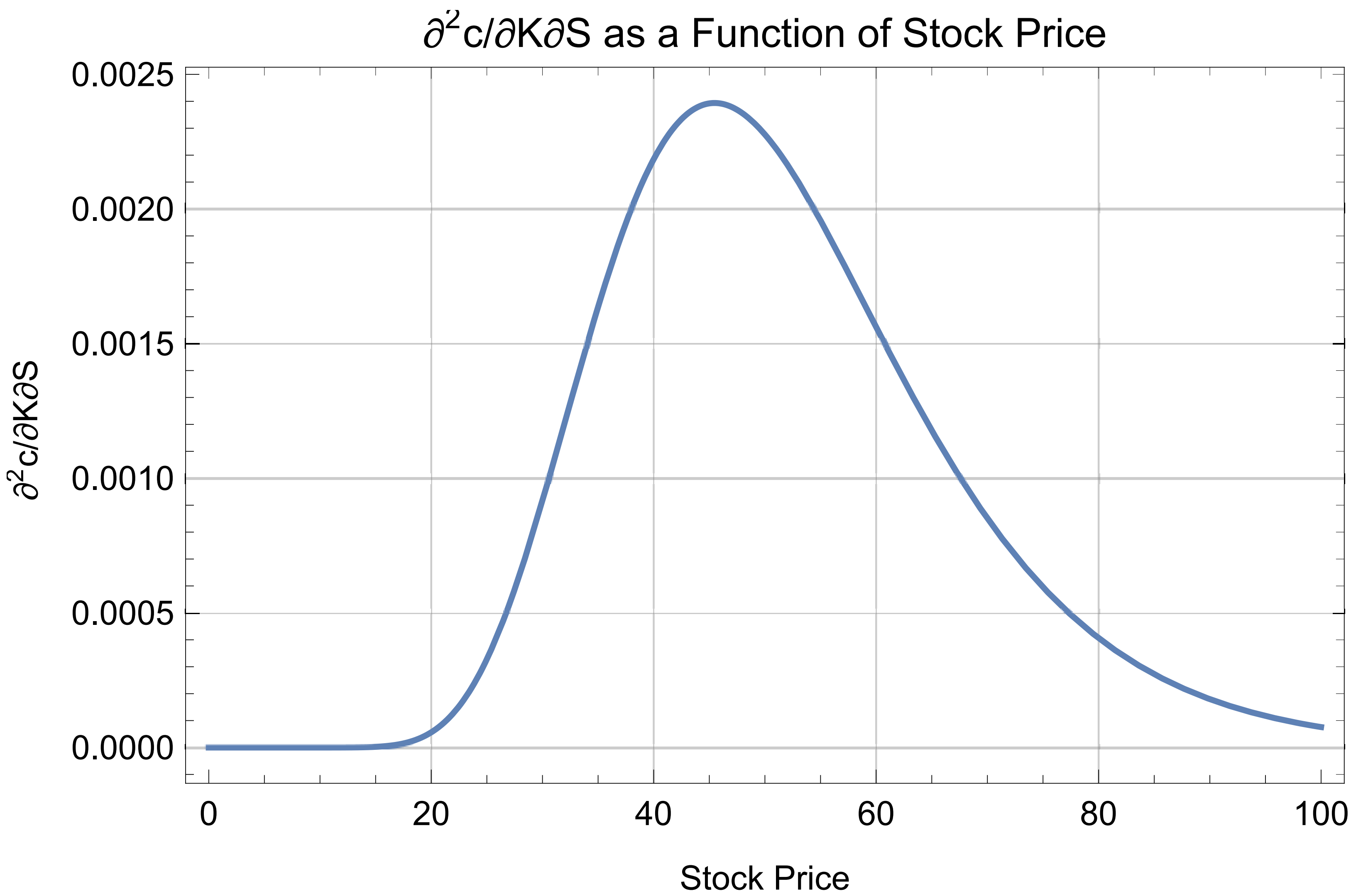


2.3 Relationship between $\frac{\partial c}{\partial K}$, $\frac{\partial p}{\partial K}$ and price of underlying

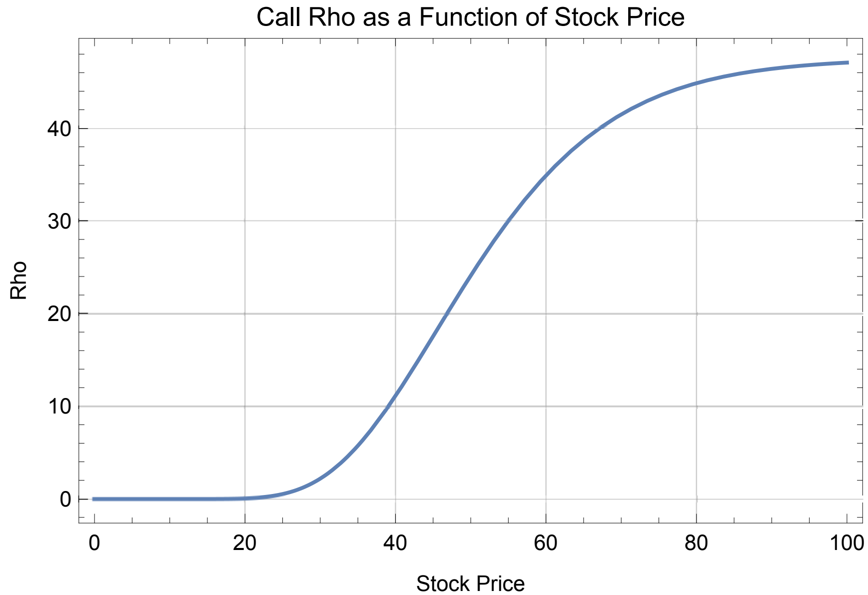


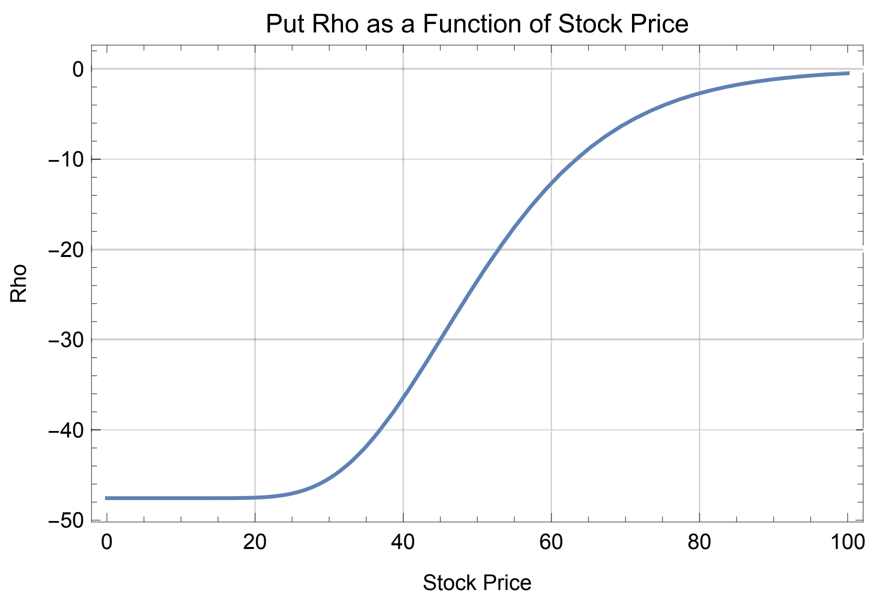


Since $\dfrac{\partial^2 c}{\partial K \partial S} = \dfrac{\partial^2 p}{\partial K \partial S} = e^{-r\tau} n(d_2)/S\sigma\sqrt{\tau}$, we can draw the graph for this "gamma" as follows:

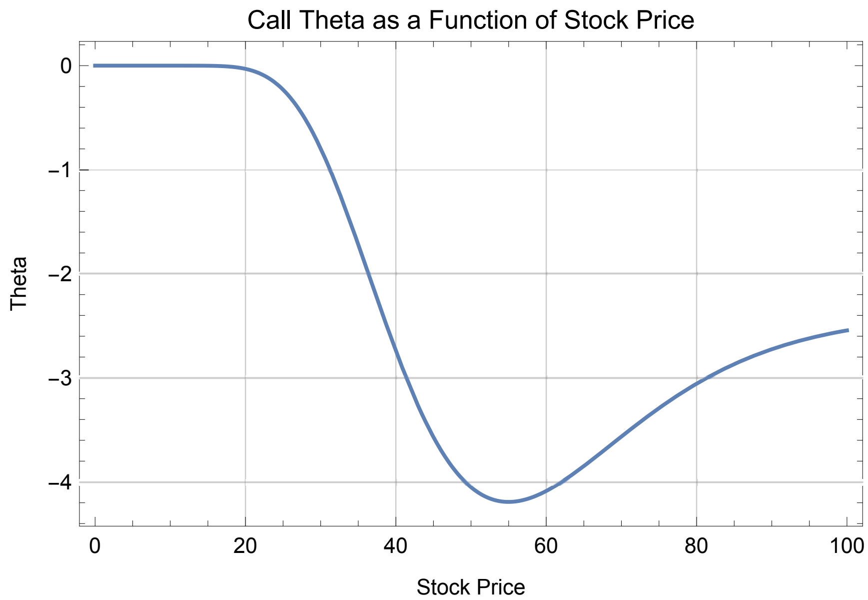


2.4 Relationship between Option Rhos and price of underlying

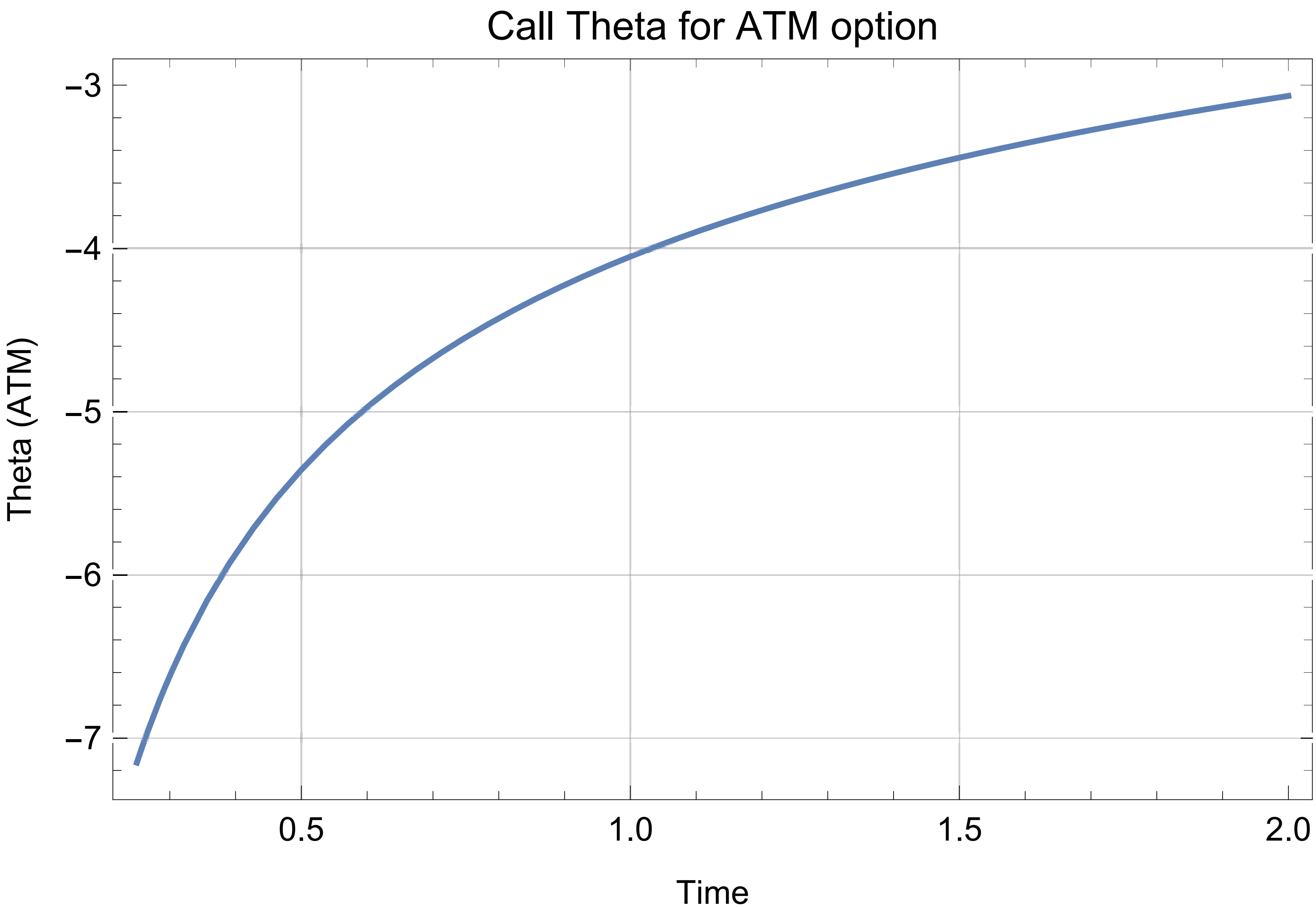




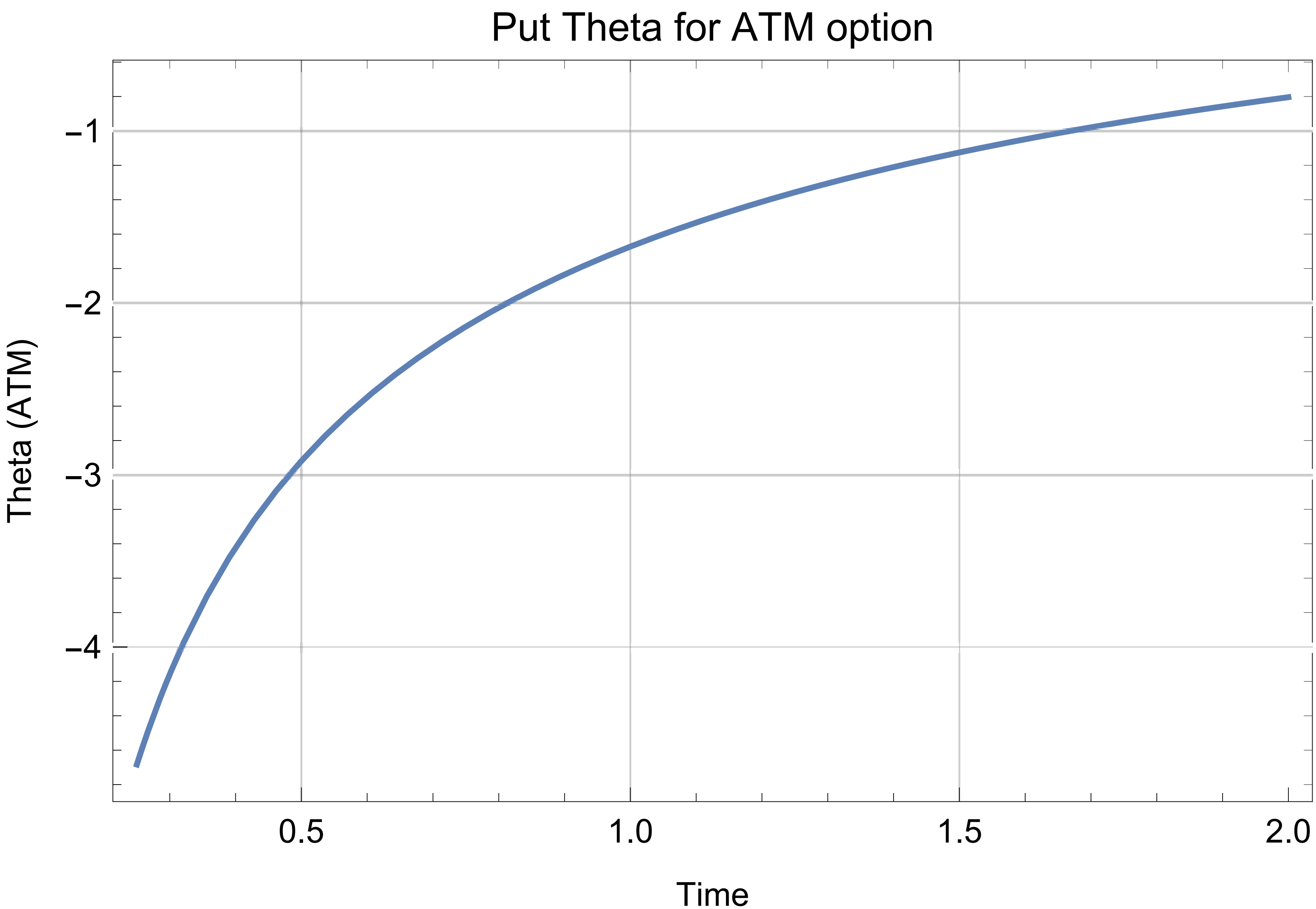
2.5 Relationship between Option Thetas and price of underlying



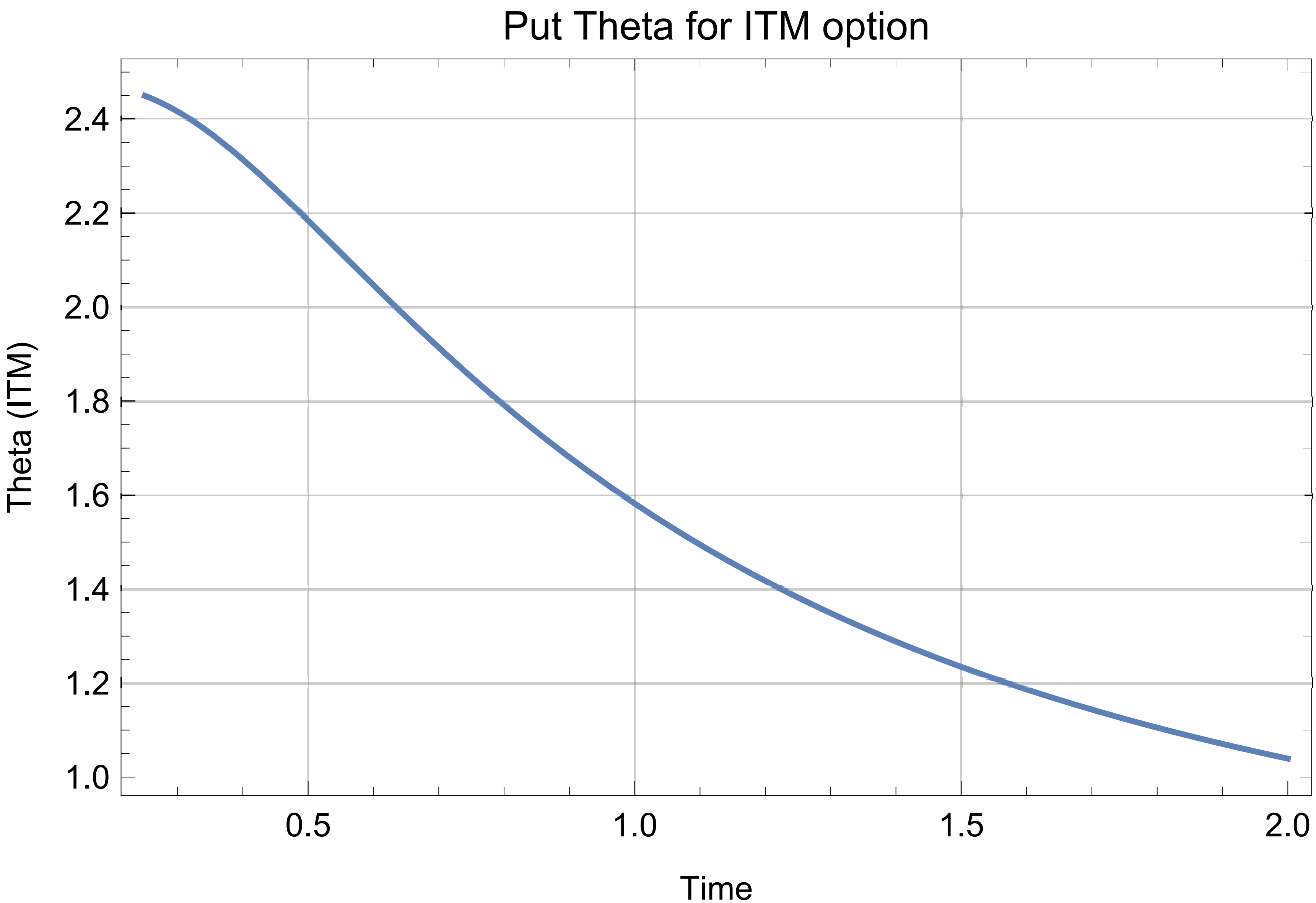
Rate of time decay (as time passes) for at-the-money call:



Rate of time decay (as time passes) for at-the-money put:



Rate of time decay (as time passes) for in-the-money put ($K = 30$):



2.6 Relationship between Option Vegas and the price of the underlying