More Wiener Processes Practice Problems (Solutions)

Problem 1.
A company's cash position, measured in millions of dollars, follows a generalized Wiener process with a drift rate of 0.5 per quarter and a variance rate of 4.0 per quarter. How high does the company's initial cash position have to be for the company to have a less than 5% chance of a negative cash position by the end of one year?

Suppose that the company's initial cash position is x. The probability distribution of the cash position at the end of one year is

$$N(x + 4 \times 0.5, 4 \times 4) = N(x + 2.0,\ 16)$$

where $N(m, v)$ is a normal probability distribution with mean m and variance v. The probability of a negative cash position at the end of one year is

$$N\left(-\frac{x + 2.0}{4}\right)$$

where $N(x)$ is the cumulative probability that a standardized normal variable (with mean zero and standard deviation 1.0) is less than x. From the properties of the normal distribution

$$N\left(-\frac{x + 2.0}{4}\right) = 0.05$$

when:

$$-\frac{x + 2.0}{4} = -1.6449$$

i.e., when $x = 4.5796$. The initial cash position must therefore be \$4.58 million.

Problem 2.
Consider a variable, Y, that follows the process

$$dY = \mu\, dt + \sigma\, dz$$

For the first three years, $\mu = 2$ and $\sigma = 3$; for the next three years, $\mu = 3$ and $\sigma = 4$. If the initial value of the variable is 5, what is the probability distribution (i.e., mean and variance) of the value of the variable at the end of year six? At the end of year six, what is the probability that $Y < 0$?

The change in Y during the first three years has the probability distribution

$$N(2 \times 3,\ 9 \times 3) = N(6,\ 27)$$

The change during the next three years has the probability distribution

$$N(3 \times 3,\ 16 \times 3) = N(9,\ 48)$$

The change during the six years is the sum of a variable with probability distribution $N(6,\ 27)$ and a variable with probability distribution $N(9,\ 48)$. The probability distribution of the change is therefore

$$N(6 + 9, 27 + 48)$$

$$= N(15,\ 75)$$

Since the initial value of the variable is 5, the probability distribution of the value of the variable at the end of year six is

$$N(20,\ 75)$$

Given this probability distribution, $z = \dfrac{(0 - 20)}{\sqrt{75}} = -2.31 \Rightarrow N(-2.31) = 1.04\%$.

Problem 3.

A stock price is currently \$80. Its expected return and volatility are 8% and 25%, respectively. What is the probability that the stock price will be greater than \$100 in two years? (Hint $S_T > 100$ when ln $S_T >$ ln 100.)

The variable $\ln S_T$ is normally distributed with mean $\ln S_0 + (\mu - \sigma^2/2)T$ and standard deviation $\sigma\sqrt{T}$. In this case $S_0 = 80$, $\mu = 0.08$, $T = 2$, and $\sigma = 0.25$ so that the mean and standard deviation of $\ln S_T$ are $\ln 80 + (0.08 - 0.25^2/2)2 = 4.480$ and $0.25\sqrt{2} = 0.354$, respectively. Also, $\ln 100 = 4.605$. The probability that $S_T > 100$ is the same as the probability that $lnS_T > 4.605$. This is

$$1 - N\left(\frac{4.605 - 4.480}{0.354}\right) = 1 - N(0.353)$$

where $N(x)$ is the probability that a normally distributed variable with mean zero and standard deviation 1 is less than x. From the tables at the back of the book $N(0.353) = 0.638$ so that the required probability is 0.362.