Baylor University
Hankamer School of Business
Department of Finance, Insurance & Real Estate

Options, Futures, and Other Derivatives
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Sample Midterm 2 Exam

Name: SOLUTIONS

Notes:

1. This test consists of 3 problems worth 32 points each. Since the maximum number of points possible is 96, I will award everyone who takes this exam an extra 4 points; thus the maximum number of points on this exam is 100. :-)
2. You may have the entire class period in order to complete this examination. Be sure to show your work as well as provide a complete answer for each problem; i.e., in addition to producing numerical results, also explain your results in plain English.

Good luck!

Problem 1 (32 points)

Assume that the ABC stock price over the next three months is described by a three-period binomial model with u =1.05 and d =0.95. Each period represents one month. The (annualized) riskless rate of interest is 5%. Assume that ABC stock is trading at \$100 per share now and that ABC will not pay any dividends during the course of the next three months.

A. (8 points) What is today's price of a European call option on ABC stock that expires in three months and has an exercise price of $K = \$100$?

   SOLUTION: There are a number of ways to solve this problem; e.g., the delta hedging, replicating portfolio, or risk neutral valuation approaches can be applied in conjunction with backward induction. However, since the option is European and may only be exercised after three time-steps, the simplest method is to determine nodes after three time-steps in which the call is in the money and then apply risk neutral valuation to pricing the option payoffs at these nodes; this is commonly referred to as the "Cox-Ross-Rubinstein", or CRR approach).

   Under CRR, the minimum number of up moves required for the call option to expire in-the-money is a, where a corresponds to the smallest integer value $> \ln(K/Sd^n)/\ln(u/d)$. Since the right-hand side of this inequality equals $\ln(100/100.95^3)/\ln(1.05/.95) = 1.54$, it follows that this option will expire in-the-money after two up moves (i.e., at node uud, and after three up moves (i.e., at node uuu).

   Since $S_{uuu} = u^3S = 1.05^3(100) = \$115.76$ and $S_{uud} = u^2dS = 1.05^2(.95)(100) = \$104.74$, given that $K = \$100$, it follows that $c_{uuu} = \$15.76$ and $c_{uud} = \$4.74$. Furthermore, since the risk neutral probability of an up move is $q = \dfrac{e^{r\delta t} - d}{u - d} = \dfrac{e^{.05/12} - .95}{1.05 - .95} = .5418$, the risk neutral probability of arriving at node uuu is q^3, whereas the risk neutral probability of arriving at node uud is $3q^2(1-q)$. Consequently, today's "arbitrage-free" price for this call option is:

   $$c = e^{-r3\delta t}[q^3c_{uuu} + 3q^2(1-q)c_{uud}] = e^{-.05/4}[(.5418^3)(15.76) + 3(.5418^2)(.4582)(4.74)] = \$4.36.$$

B. (8 points) What is today's price of a European put option on ABC stock that expires in three months and has an exercise price of $K = \$100$?

   SOLUTION: Since we know the prices for the call, the riskless bond, and the share, we can infer the put price by applying the put-call parity equation:

   $$c + Ke^{-r3\delta t} = p + S \Rightarrow p = c + Ke^{-r3\delta t} - S = 4.36 + 100e^{-.05/4} - 100 = \$3.12.$$

C. (16 points) Solve for the arbitrage-free European call option prices at nodes uu, ud, dd, u, d, and the beginning of the tree via backward induction.

   SOLUTION: From part A of this problem, we know that $c_{uuu} = \$15.76$ and $c_{uud} = \$4.74$. We can also infer that $c_{udd} = c_{ddd} = \$0$.

Applying risk neutral valuation in conjunction with backward induction, we find the arbitrage-free European call option prices at nodes uu, ud, dd, u, d, and the beginning of the tree

$$c_{uu} = e^{-r\delta t}[qc_{uuu} + (1-q)c_{uud}] = e^{-.05}[.5418(15.76) + .4582(4.74)] = \$10.67.$$
$$c_{ud} = e^{-r\delta t}[qc_{uud} + (1-q)c_{udd}] = e^{-.05}[.5418(4.74)] = \$2.56.$$
$$c_{dd} = e^{-r\delta t}[qc_{uud} + (1-q)c_{udd}] = \$0.$$
$$c_{u} = e^{-r\delta t}[qc_{uu} + (1-q)c_{ud}] = e^{-.05}[.5418(10.67) + .4582(2.56)] = \$6.92.$$
$$c_{d} = e^{-r\delta t}[qc_{ud} + (1-q)c_{dd}] = e^{-.05}[.5418(2.56)] = \$1.38.$$
$$c = e^{-r\delta t}[qc_{u} + (1-q)c_{d}] = e^{-.05}[.5418(6.92) + .4582(1.38)] = \$4.36.$$

Problem 2 (32 points)

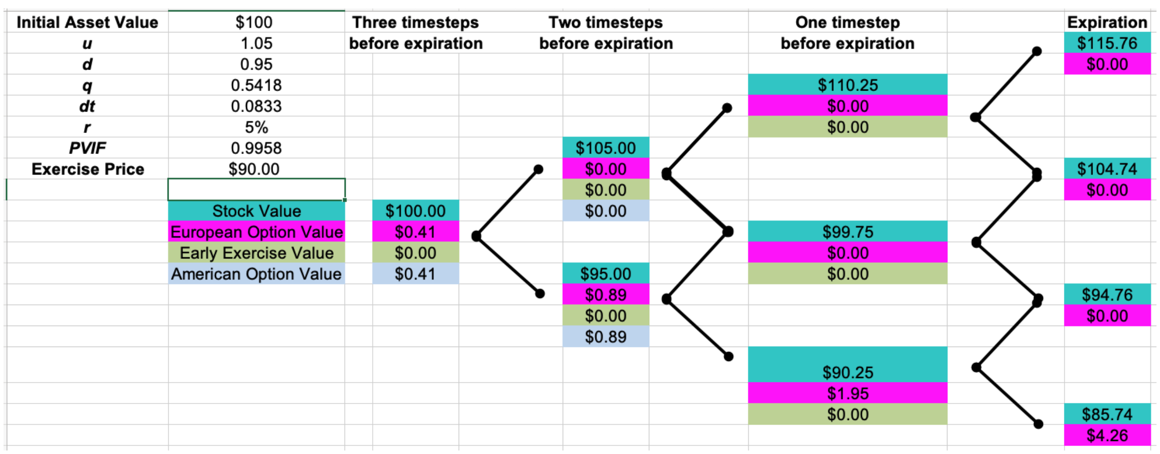
Using the information from problem 1, calculate the values of the following three options:

A. (10 points) An American put option with an exercise price of \$90.

SOLUTION: By inspection of the stock binomial tree, the only state in which the put would ever be in-the-money would be in the *ddd* state. In that state, since $S_{ddd} = \$85.74$, it follows that $p_{ddd} = \$4.26$. Applying risk neutral valuation, $p_{dd} = e^{-r\delta t}[qp_{udd} + (1-q)p_{ddd}]$ $= e^{-.05/4}[.4582(4.26)] = \$1.95$. However, at node *dd*, it does not pay to exercise early since the exercised value of the option is $Max(0, K - S_{dd}) = Max(0, 90 - 90.25) = \$0$. Similarly, the exercised value of the put is \$0 at all other nodes. Since it never pays to exercise the this American put option early, it follows that its value is the same as a European put with an exercise price of \$90. Therefore, the "arbitrage-free" price for this American put option is:

$$p = e^{-r3\delta t}[(1-q)^3 p_{ddd}] = e^{-.05/4}[(.4582^3)(4.26)] = \$0.41.$$

The following figure shows how the binomial tree for a three-month American put option with an exercise price of \$90 evolves over time, and why its arbitrage-free price is the same as the arbitrage-free price for an otherwise identical European put option:



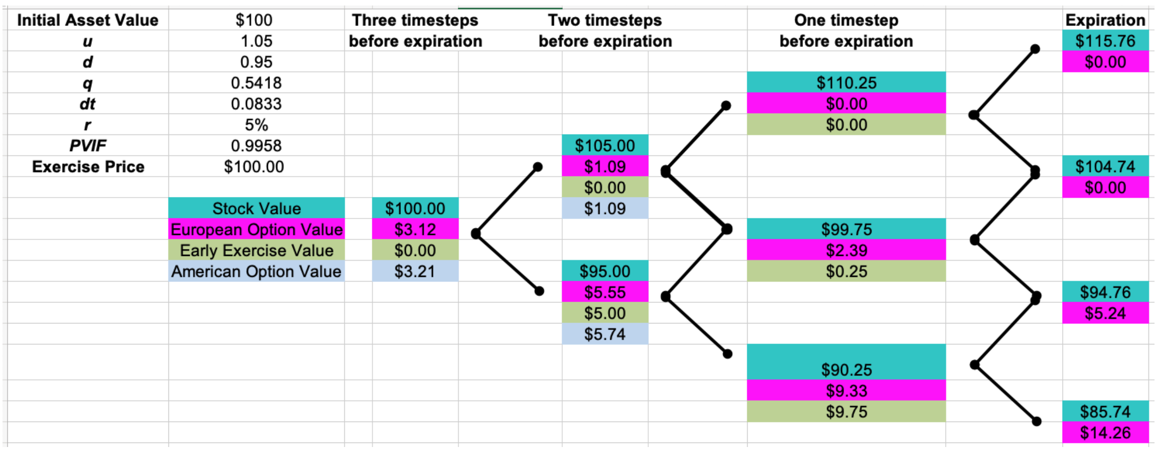
B. (11 points) An American put option with an exercise price of \$100.

SOLUTION: You clearly would not exercise today, since the option is at-the-money at inception. By inspection of the stock binomial tree, it also is not worthwhile to exercise early at nodes *u* and *uu*. Early exercise at node *ud* would provide a payoff of \$0.25, but it doesn't make sense to exercise at that node since by *not* exercising, in one month the payoffs at nodes *uud* and *udd* are [\$0, \$5.24], implying a market value at node *ud* of $p_{ud} = e^{-r\delta t}[qp_{uud} + (1-q)p_{udd}] = e^{-.05/12}[.5418(0) + .4582(5.24)] = \$2.39$.

Similarly, you would not exercise early at node *d* since by *not* exercising, the node *ud* and *dd* put prices are $p_{ud} = \$2.39$ and $p_{dd} = e^{-.05/12}[.5418(5.24) + .4582(14.26)] = \$9.33$, implying a market value at node d of $p_d = e^{-.05/12}[.5418(2.39) + .4582(9.33)] = \$5.55$ compared with an early exercise value of $Max(0, K - S_d) = Max(0, 100 - 95) = \$5$.

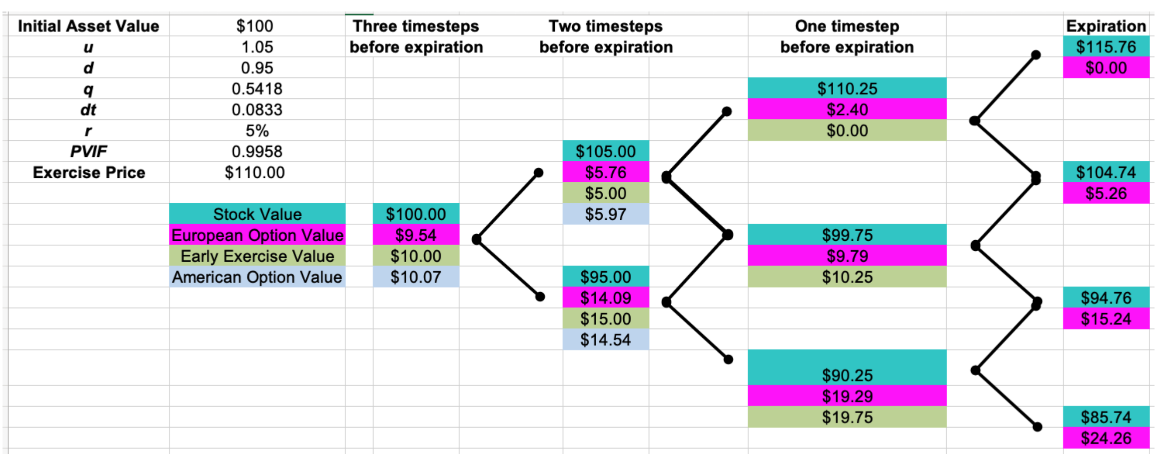
However, it would make sense to exercise early at node dd, since $Max(0, K - S_{dd}) = \$9.75 > p_{dd} = \$9.33$. Thus, the value of the American put option with an exercise price of \$100 at node d is worth $p_d = e^{-.05/12}[.5418(2.39) + .4582(9.75)] = \$5.74$, and at node u, it is worth $p_u = e^{-.05/12}[.5418(0) + .4582(5.24)] = \$1.09$, which in turn implies that the current market value of an American put option with an exercise price of \$100 is worth $p = e^{-.05/12}[.5418(1.09) + .4582(5.74)] = \$3.21$.

The following figure shows how the binomial tree for a three-month American put option with an exercise price of \$100 evolves over time, and why its arbitrage-free price exceeds the arbitrage-free price for an otherwise identical European put option by 9 cents:



C. (11 points) An American put option with an exercise price of \$110.

The following figure shows how the binomial tree for a three-month American put option with an exercise price of \$110 evolves over time:



SOLUTION: By inspection, a European put option with an exercise price of \$110 is in the money at nodes uud, udd, and ddd; i.e., $p_{uud} = \$5.26, p_{udd} = \$15.24$, and

$p_{ddd} = \$24.26$. Applying risk neutral valuation, a European put option with an exercise price of \$110 is worth the following amount today:

$$p = e^{-r3\delta t}[3q^2(1-q)p_{uud} + 3q(1-q)^2 p_{udd} + (1-q)^3 p_{udd}]$$
$$= e^{-.05/4}[3(.5418^2)(.4582)(5.26) + 3(.5418)(.4582^2)(15.24) + (.4582^3)(24.26)] = \$9.54.$$

Since the exercised value of the American put at the tree's inception is $Max(0, K-S) = Max(0, 110-100) = \$10$, immediate exercise might be considered. However, upon closer inspection, since early exercise at node ud is worth \$10.25 compared with \$9.79 for retaining the option, the American put at node u is worth \$5.97. Furthermore, early exercise is optimal at node d since it yields a \$15 value compared with the American put value of \$14.54. Discounting \$5.97 and \$15 back to the beginning of the tree, we find that the American put is worth \$10.07, which exceeds the immediate exercise value by 7 cents.

Problem #3 (32 points)

Consider a European call option on a non-dividend-paying stock where the stock price is \$90, the exercise price is \$115, the (annualized) risk-free rate is $r = 3\%$, the (annualized) volatility is $\sigma = 30\%$ per year, $u = e^{\sigma\sqrt{\delta t}}$, $d = 1/u$, the number of timesteps is 4, and the time to expiration is 1 year.

A. (16 points) What is the arbitrage-free price for this call option?

SOLUTION: The easiest method for finding the price of this call option is to apply the Cox-Ross-Rubinstein binomial option pricing model. The first step involves determining the minimum number of up moves (denoted as "a") required in order for the call option to expire in-the-money; since a is the smallest integer value $> \ln(K/Sd^n)/\ln(u/d)$ and $\ln(K/Sd^n)/\ln(u/d) = \ln(115/90(e^{4(-.3\sqrt{.25})})/\ln(e^{2(.3\sqrt{.25})}) = 2.8171$, this implies that $a = 3$; thus, the call option will be in the money after 3 and 4 up moves (i.e., at nodes $uuud$ and $uuuu$). Thus,

$$c_{uuuu} = \max[0, u^4S - K] = 1.1618^4(90) - 115 = \$48.99; \text{ and}$$
$$c_{uuud} = \max[0, u^3dS - K] = 1.1618^3(.8607)(90) - 115 = \$6.49.$$

The risk neutral probability of one up move is

$$q = \frac{e^{r\delta t} - d}{u - d} = \frac{e^{r\delta t} - e^{-\sigma\sqrt{\delta t}}}{e^{\sigma\sqrt{\delta t}} - e^{-\sigma\sqrt{\delta t}}}$$
$$= \frac{e^{.03(.25)} - e^{-.3\sqrt{.25}}}{e^{.3\sqrt{.25}} - e^{-3\sqrt{.25}}} = .4876.$$

Applying risk neutral valuation, we find that

$$\begin{aligned} c &= e^{-4r\delta t}[q^4c_{uuuu} + 4q^3(1-q)c_{uuud}] \\ &= e^{-.03}[.4876^4(48.99) + 4(.4876^3)(.5124)(6.49)] \\ &= \$4.18. \end{aligned}$$

B. (16 points) What is the current market value of an otherwise identical (i.e., same underlying asset, same strike price, interest rate, same volatility, same number of timesteps, and time to expiration) European put option?

SOLUTION: Applying the put-call parity equation, $p = c + Ke^{-rn\delta t} - S = \$4.18 + \$115e^{-.03} - \$90 = \$25.78$.