Actual vs. Risk Neutral Probability of Call Option Expiring In-The-Money

An important simplification for pricing an option in the discrete time binomial framework involves the use of the so-called "risk neutral" probability (q) of an up move.

- Since the expected value of the underlying asset price ($E(S_{\delta t})$) one δt timestep from today is $E(S_{\delta t}) = puS + (1-p)dS = e^{\mu \delta t}S$ (see pp. 20-21 of the Binomial Trees lecture note), it follows that the actual probability (p) of an up move is

$$p = (e^{\mu \delta t} - d)/(u - d),$$

  where μ corresponds to the annual (continuously compounded) expected rate of return on the underlying asset.

- We obtain the risk neutral probability q by replacing μ with r:

$$q = (e^{r \delta t} - d)/(u - d).$$

Since $\mu - r > 0$ in a risk averse world, it follows that $p > q$.

In the continuous time framework, a similar relationship exists between the actual and risk neutral probabilities of a call option expiring in the money.

- Consider the Black-Scholes-Merton equation for a call option:

$$C = SN(d_1) - Ke^{-rT}N(d_2),$$

  where $N(d_2)$ corresponds to the risk neutral probability of a call option expiring in-the-money.

- As noted on pages 8-10 of the Wiener Processes and Îto's Lemma lecture note, since the date T stock price ($\tilde{S}_T$) is *lognormally* distributed, so is its gross return $\tilde{S}_T/S$.

- By applying Îto's Lemma, we find that the continuously compounded rate of return on the stock ($\ln \tilde{S}_T/S$) is normally distributed with mean $(\mu - .5\sigma^2)T$ and variance $\sigma^2 T$; i.e., $\ln \tilde{S}_T/S \sim N((\mu - .5\sigma^2)T, \sigma^2 T)$.

Since the risk neutral probability that the call option will expire in-the-money is $N(d_2) = N\left(\frac{\ln(S/K) + (r - .5\sigma^2)T}{\sigma\sqrt{T}}\right)$, then the actual probability of expiring in-the-money is calculated by replacing the riskless rate of return r with the expected return μ in the numerator for d_2; i.e., the actual probability is $N\left(\frac{\ln(S/K) + (\mu - .5\sigma^2)T}{\sigma\sqrt{T}}\right)$.

See the "Actual versus Risk Neutral Probabilities" spreadsheet for numerical examples of actual versus risk neutral probabilities of a European call option expiring in-the-money.