Applying Itô's Lemma to determine the probability distribution parameters for the continuously compounded rate of return

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Suppose that the price of an asset changes continuously according to the Geometric Brownian Motion equation; i.e,

$$dS = \mu S dt + \sigma S dz, \tag{1}$$

In equation (1), S is the current (date t) price of an asset, μ is its annualized expected return, σ represents its annualized volatility, and dS corresponds to the instantaneous change in the asset price (i.e., the change that occurs between date t and date $t + dt$). As shown in equation (1), instantaneous asset price changes are partially *deterministic* (as indicated by the so-called "drift", or $\mu S dt$ term) and partially *stochastic* (as indicated by the $\sigma S dz$ term). The $\sigma S dz$ term is stochastic because $dz = \epsilon\sqrt{dt}$ is a Wiener process with mean $E(dz) = 0$ and variance $Var(dz) = dt$.[1]

The Geometric Brownian Motion equation is often referred to as an *exponential* stochastic differential equation because its solution is an exponential function; specifically, $S_T = Se^x$, where S_T represents the asset price $T - t$ periods from now, $T - t \geq 0$, and $x = (\mu - \sigma^2/2)(T - t) + \epsilon\sigma\sqrt{T - t}$. Since ϵ is normally distributed, so is x; thus, S_T is lognormally distributed.[2] As of time T, the mean of S_T is $E(S_T) = Se^{\mu(T-t)}$, and its variance is $\sigma^2_{S_T} = S^2e^{2\mu(T-t)}(e^{\sigma^2(T-t)} - 1) = (E(S_T))^2(e^{\sigma^2(T-t)} - 1)$. Since S_T is lognormally distributed, S_T/S

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[1] Pennacchi (2008) makes the following very important observation about Wiener processes (cf. footnote 12 on page 240): "...it may be helpful to remember that in the continuous-time limit $dz^2 = dt$, but $dzdt = 0$, and $dt^n = 0$ for $n > 1$."

[2] The lognormal distribution is a particularly suitable candidate for modeling asset prices. Besides being mathematically tractable, the lognormal distribution generates price patterns that resemble real world asset price patterns. Furthermore, under the lognormal distribution, asset prices are non-negative, being bounded from below at zero and unbounded from above.

is also lognormally distributed; therefore, ln S_T/S is normally distributed. Furthermore, ln S_T/S corresponds to the T - t period *continuously compounded* rate of return on the asset.[3]

In order to find the expected value and variance of ln S_T/S, some calculus is required. Consider a Taylor series expansion for the function $G = G(S,t)$ around date t and the value of S at date t:

$$\begin{aligned} G(S(t+dt), t+dt) &= G(S(t),t) + \frac{\partial G}{\partial t}dt + \frac{\partial G}{\partial S}dS \\ &+ \frac{1}{2}\left[\frac{\partial^2 G}{\partial t^2}dt^2 + \frac{\partial^2 G}{\partial S^2}dS^2 + \frac{\partial^2 G}{\partial S \partial t}dSdt\right] + R, \end{aligned} \tag{2}$$

where R corresponds to the "remainder" term consisting of third and higher order terms. Fortunately, since $dz^2 = dt$, $dzdt = 0$, and $dt^n = 0$ for $n > 1$ (see Pennacchi (2008), footnote 12 on page 240), the first and third bracketed terms in equation (2) vanish, resulting in equation (3):

$$G(S(t+dt), t+dt) - G(S(t),t) = dG = \frac{\partial G}{\partial t}dt + \frac{\partial G}{\partial S}dS + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}dS^2 \tag{3}$$

Equation (3) is Itô's Lemma, also known as the *fundamental theorem of stochastic calculus* (see Pennacchi (2008), p. 238). Since $dS^2 = \sigma^2 S^2 dt$,[4] we substitute this expression for dS^2 into equation (3) and obtain equation (4):

$$dG = \frac{\partial G}{\partial t}dt + \frac{\partial G}{\partial S}dS + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 G}{\partial S^2}dt \tag{4}$$

Substituting the right-hand side of equation (1) into the right-hand side of equation (4)

[3]This highlights yet another advantage of assuming that asset prices are lognormally distributed. Lognormally distributed prices imply that continuously compounded asset returns ($x = \ln(S_T/S)$) are normally distributed, thus resembling real world returns (in the sense that realized returns can be negative, zero, or positive).

[4]As noted in the previous paragraph, all terms involving dt raised to powers greater than 1 equal zero; thus, $dS^2 = (\mu S dt + \sigma S dz)^2 = \mu^2 S^2 dt^2 + \sigma^2 S^2 dt + 2\mu\sigma S dt^{3/2} = \sigma^2 S^2 dt$.

yields

$$dG = \left(\frac{\partial G}{\partial t} + \frac{\partial G}{\partial S}\mu S + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 G}{\partial S^2} \right) dt + \frac{\partial G}{\partial S}\sigma S dz \tag{5}$$

Suppose $G = \ln S$. Then $\frac{\partial G}{\partial S} = \frac{1}{S}$, $\frac{\partial^2 G}{\partial S^2} = -\frac{1}{S^2}$, and $\frac{\partial G}{\partial t} = 0$. Therefore,

$$\begin{aligned} dG &= \left(0 + \frac{1}{S}\mu S + \frac{1}{2}\sigma^2 S^2 \left(-\frac{1}{S^2} \right) \right) dt + \frac{1}{S}\sigma S dz \\ &= \left(\mu - \frac{1}{2}\sigma^2 \right) dt + \sigma dz. \end{aligned} \tag{6}$$

Consequently, the change in the natural logarithm of the asset price between now (date 0) and date T, $\ln S_T - \ln S = \ln \frac{S_T}{S}$, is normally distributed with mean $\left(\mu - \frac{1}{2}\sigma^2 \right) T$ and variance $\sigma^2 T$; i.e.,

$$\ln S_T - \ln S \sim N\left(\left(\mu - \frac{1}{2}\sigma^2 \right) T, \sigma^2 T \right) \tag{7}$$

or equivalently,

$$\ln S_T \sim N\left(\ln S + \left(\mu - \frac{1}{2}\sigma^2 \right) T, \sigma^2 T \right) \tag{8}$$

Note that the expressions given by (7) and (8) correspond to expressions (18) and (19) in the "Wiener Processes and Ito's Lemma" textbook chapter.

References

Pennacchi, G. G. (2008): *Theory of Asset Pricing*. Pearson/Addison-Wesley Boston.