Wiener Processes and Ito's Lemma

Modeling Stock Prices

- We have previously modeled stock prices in a binomial (discrete time) framework.
- Here, we show (among other things) that the limiting case of the binomial tree (i.e., when the timestep $\delta t \to 0$) yields the continuous time model.

Markov Processes

- In a Markov process, future movements in a random variable depend only on *where we are*, not the history of how we *got where we are*; i.e., there is no serial correlation.
- This is also known as a "random walk".
- We assume in both the discrete and continuous time cases that stock prices follow Markov processes.

Discrete & Continuous Stock Return Dynamics

- Since changes in asset prices over time (notated as $\delta S/S$) follow Markov processes, the following "time series" equation for $\delta S/S$ is implied:

$$\delta S / S = \mu \delta t + \sigma \varepsilon \sqrt{\delta t},$$

where ε is a standard normal random variable; i.e., $\varepsilon \sim N(0,1)$.

Discrete & Continuous Stock Return Dynamics

- Note that $E(\delta S / S) = E\left(\mu\delta t + \sigma\varepsilon\sqrt{\delta t}\right) = \mu\delta t$,

  since $E(\varepsilon) = 0.$

- Also note that

$$\sigma_{\delta S/S}^2 = E\left[\left(\delta S / S - \mu\delta t\right)^2\right]$$

$$= E\left[\left(\mu\delta t + \sigma\varepsilon\sqrt{\delta t} - \mu\delta t\right)^2\right]$$

$$= \sigma^2\delta t E\left[\varepsilon^2\right] = \sigma^2\delta t.$$

Wiener Process

- In the equation $\delta S / S = \mu \delta t + \sigma \varepsilon \sqrt{\delta t}$, the second term ($\sigma \varepsilon \sqrt{\delta t}$) is typically written $\sigma \delta z$.
- δz is commonly referred to as a *Wiener process* (so named in honor of American mathematician Norbert Wiener (1894-1964)).
- The Wiener process has been used in physics to describe the motion of a particle that is subject to a large number of small molecular shocks, and is frequently referred to as Brownian motion.

Geometric Brownian Motion (GBM)

- The continuous time analog of the discrete time return equation is the geometric Brownian motion equation:

$$dS / S = \mu dt + \sigma dz,$$

where $dz = \varepsilon\sqrt{dt}$.

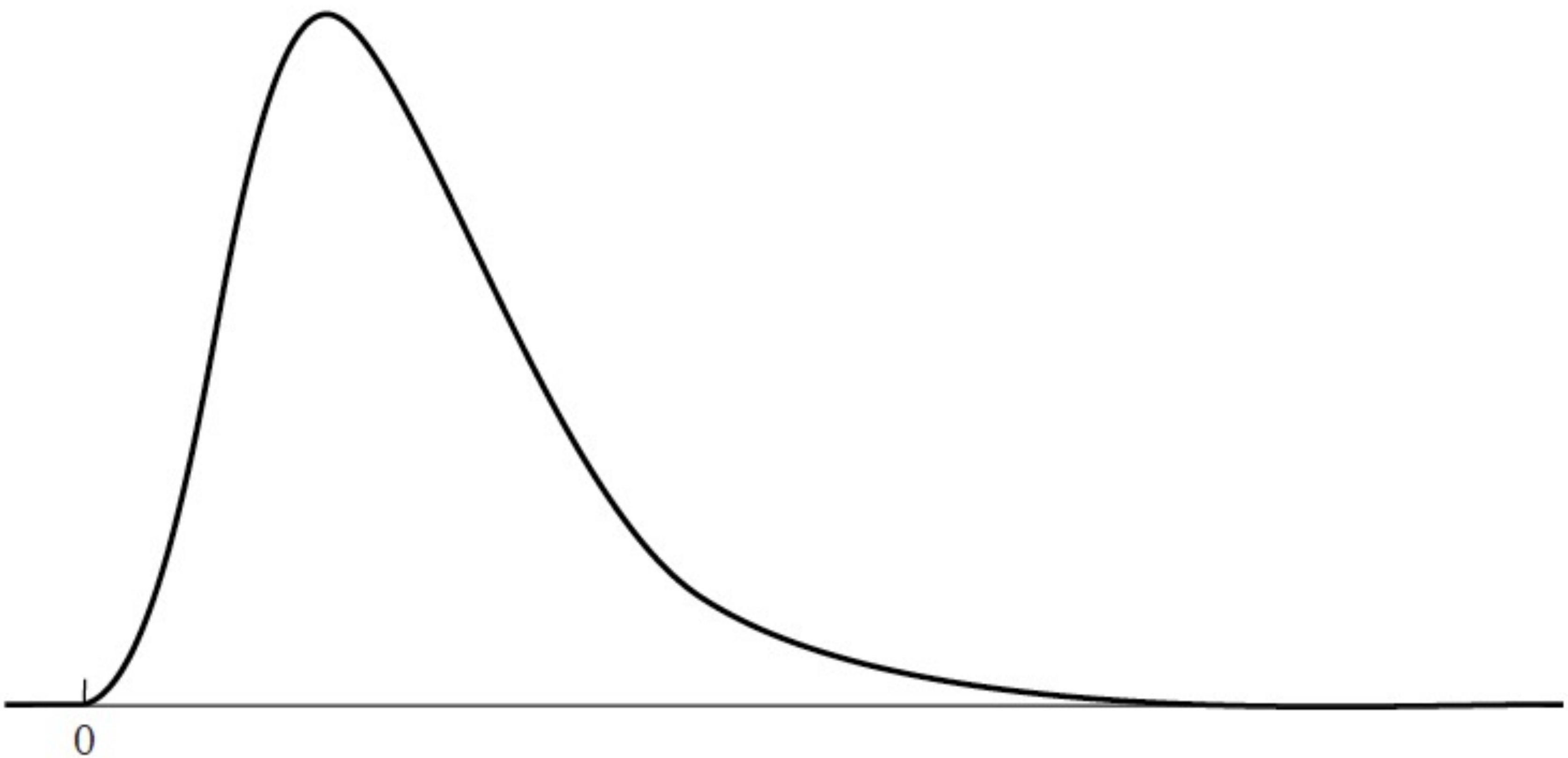
- Thus, asset price changes conform to the following equation:

$$dS = \mu S dt + \sigma S dz. \qquad (1)$$

Geometric Brownian Motion (GBM)

- Equation (1) implies that asset prices are lognormally distributed; thus, they are bounded from below at 0, and unbounded from above.
- The mean and variance of the lognormally distributed asset price T periods from today are $S_T = Se^{\mu T}$, and $\sigma^2_{S_T} = S^2 e^{2\mu T}(e^{2\sigma^2 T} - 1)$. Suppose that $S = \$20$, $T = 1$, $\mu = .20$, and $\sigma = .4$. Then $E(S(T)) = 20e^{.2(1)} = \$24.43$, $\sigma^2_{S(T)} = 20^2 e^{2(.2(1))}(e^{(.04^2(1))} - 1) = 103.54$, and $\sigma_{S(T)} = \sqrt{103.54} = 10.18$.

The Lognormal Distribution



Ito's Lemma

- Since $\tilde{S}_T / S = \tilde{R}_T$ is a *lognormally* distributed random variable, it follows that the (continuously compounded) log return $\ln \tilde{S}_T / S = \ln \tilde{R}_T$ is *normally* distributed.
- In order to determine the mean and standard deviation for the log return distribution, some calculus is required.
- However, since we know that asset prices evolve according to the Geometric Brownian Motion equation, we can use Ito's Lemma to make this determination!

Ito's Lemma

Let $G = G(S, t)$. Then according to Ito's Lemma,

$$dG = \frac{\partial G}{\partial t}dt + \frac{\partial G}{\partial S}dS + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}dS^2 . \qquad (3)$$

Substituting $dS^2 = S^2\sigma^2 dt$ into (3) yields equation (4):

$$dG = \frac{\partial G}{\partial t}dt + \frac{\partial G}{\partial S}dS + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 G}{\partial S^2}dt . \qquad (4)$$

Substituting the right-hand side of equation (1) into the right-hand side of equation (4) yields

$$dG = \left(\frac{\partial G}{\partial t} + \frac{\partial G}{\partial S}\mu S + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 G}{\partial S^2}\right)dt + \frac{\partial G}{\partial S}\sigma S dz \qquad (5)$$

Ito's Lemma: First Application

Suppose $G = \ln S$. Then $\dfrac{\partial G}{\partial S} = \dfrac{1}{S}$,

$\dfrac{\partial^2 G}{\partial S^2} = -\dfrac{1}{S^2}$, and $\dfrac{\partial G}{\partial t} = 0$. Therefore,

$$\begin{aligned} dG &= \left(0 + \frac{1}{S}\mu S + \frac{1}{2}\sigma^2 S^2\left(-\frac{1}{S^2}\right)\right)dt + \frac{1}{S}\sigma S\,dz \\ &= \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma\,dz. \end{aligned} \tag{6}$$

Ito's Lemma: First Application

Consequently, the change in ln S between date zero and some future date T, $\ln \tilde{S}_T - \ln S_0 = \ln \frac{\tilde{S}_T}{S_0}$ is normally distributed with mean $\left(\mu - \frac{1}{2}\sigma^2\right)T$ and variance $\sigma^2 T$;

i.e.,

$$\ln \frac{\tilde{S}_T}{S_0} \sim N\left(\left(\mu - \frac{1}{2}\sigma^2\right)T, \sigma^2 T\right), \qquad (7)$$

or equivalently,

$$\ln \tilde{S}_T \sim N\left(\ln S_0 + \left(\mu - \frac{1}{2}\sigma^2\right)T, \sigma^2 T\right). \qquad (8)$$

Intuition about μ and $\mu-\sigma^2/2$

- μ corresponds to the expected return in a very short time, δt, expressed with a compounding frequency of δt (AKA the *arithmetic* mean return).
- $\mu - \sigma^2/2$ corresponds to the expected return in a long period of time expressed with continuous compounding (AKA the *geometric* mean return).

Numerical Example

- Suppose that returns in successive years are 15%, 20%, 30%, −20% and 25% (ann. comp.)
- The arithmetic mean of the returns is 14%
- The returned that would actually be earned over the five years (the geometric mean) is 12.4% (ann. comp.)
- The arithmetic mean of 14% is analogous to μ.
- The geometric mean of 12.4% is analogous to $\mu - \sigma^2/2$.

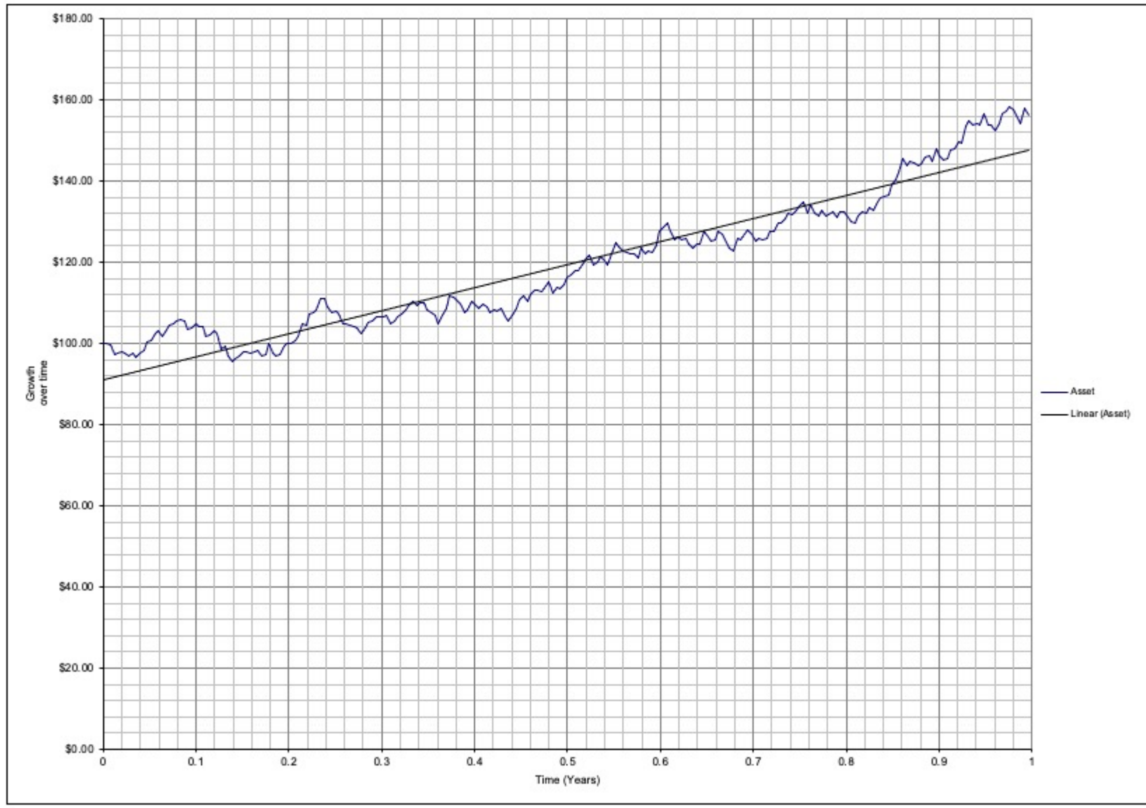
Simulation of GBM

Time	Drift	Uncertainty	Change	Asset	Return
0				$100.00	
0.004	$0.04	-$0.06	-$0.02	$99.98	-0.02%
0.008	$0.04	-$0.52	-$0.48	$99.50	-0.48%
0.012	$0.04	-$2.18	-$2.14	$97.36	-2.15%
0.016	$0.04	$0.27	$0.30	$97.66	0.31%
0.02	$0.04	$0.29	$0.33	$97.99	0.34%
0.024	$0.04	-$0.40	-$0.36	$97.63	-0.37%
0.028	$0.04	-$0.79	-$0.75	$96.88	-0.77%
0.032	$0.04	$0.59	$0.63	$97.51	0.65%
0.036	$0.04	-$1.08	-$1.04	$96.47	-1.07%
0.04	$0.04	$1.18	$1.22	$97.69	1.26%
0.044	$0.04	$0.72	$0.75	$98.44	0.77%
0.048	$0.04	$2.03	$2.07	$100.51	2.11%
0.052	$0.04	$0.05	$0.09	$100.60	0.09%
0.056	$0.04	$1.94	$1.98	$102.58	1.97%
0.06	$0.04	$0.58	$0.62	$103.20	0.61%
0.063	$0.04	-$1.50	-$1.45	$101.75	-1.41%
0.067	$0.04	$0.90	$0.94	$102.68	0.92%
0.071	$0.04	$1.89	$1.93	$104.62	1.88%
0.075	$0.04	$0.10	$0.14	$104.75	0.13%
0.079	$0.04	$0.60	$0.64	$105.39	0.61%
0.083	$0.04	$0.48	$0.53	$105.92	0.50%
0.087	$0.04	-$0.51	-$0.47	$105.45	-0.44%
0.091	$0.04	-$1.98	-$1.94	$103.51	-1.84%
0.095	$0.04	$0.36	$0.40	$103.92	0.39%
0.099	$0.04	$0.74	$0.78	$104.70	0.75%
0.103	$0.04	-$0.52	-$0.48	$104.21	-0.46%
0.107	$0.04	-$0.01	$0.03	$104.24	0.03%
0.111	$0.04	-$2.68	-$2.64	$101.60	-2.53%
0.115	$0.04	$0.28	$0.32	$101.92	0.31%
0.119	$0.04	$0.98	$1.02	$102.95	1.00%
0.123	$0.04	-$0.53	-$0.49	$102.46	-0.47%
0.127	$0.04	-$3.78	-$3.74	$98.72	-3.65%
0.131	$0.04	$0.55	$0.59	$99.31	0.60%
0.135	$0.04	-$2.29	-$2.25	$97.06	-2.26%
0.139	$0.04	-$1.58	-$1.54	$95.53	-1.58%
0.143	$0.04	$0.81	$0.84	$96.37	0.88%
0.147	$0.04	$0.63	$0.66	$97.03	0.69%
0.151	$0.04	$0.79	$0.83	$97.86	0.85%

Asset	**100**
μ	**0.1**
σ	**0.2**
Timestep	**0.004**

Daily Arithmetic Return	0.18%
Annual Arithmetic Return	46.61%
Annual Log Return	44.63%
Std. Dev.	1.19%
Min	-3.65%
25th pct.	-0.65%
Median	0.28%
75th pct.	0.96%
Max	3.09%

Daily mu - .5var	0.18%
Annual mu - .5var	44.84%
Initial price x exp(o2)	$156.57
Final simulated price	**$156.25**



Ito's Lemma: Second Application

- Recall that the application of Ito's lemma for an arbitrary function $G = G(S,t)$ gave rise to the following stochastic equation:

$$dG = \left(\frac{\partial G}{\partial t} + \frac{\partial G}{\partial S}\mu S + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 G}{\partial S^2} \right) dt + \frac{\partial G}{\partial S}\sigma S dz.$$

- Consider a forward contract on a non-dividend paying stock; its date t "arbitrage-free" price is $F(S,t) = F_t = S_t e^{r(T-t)}$.
- Next, apply the equation for dG to determine dF's equation:

$$dF = \left(\frac{\partial F}{\partial t} + \frac{\partial F}{\partial S}\mu S + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} \right) dt + \frac{\partial F}{\partial S}\sigma S dz.$$

Since $\frac{\partial F}{\partial S} = e^{r(T-t)}$, $\frac{\partial^2 F}{\partial S^2} = 0$, and $\frac{\partial F}{\partial t} = -rSe^{r(T-t)}$, then

$$dF = \left(-rSe^{r(T-t)} + e^{r(T-t)}\mu S + 0 \right) dt + e^{r(T-t)}\sigma S dz.$$

- Substituting F for $Se^{r(T-t)}$ and simplifying further, we obtain $dF = (\mu - r)F dt + \sigma F dz.$