Pricing Forwards and Futures

This lecture explores pricing for forward and futures contracts. We highlight their similarities and derive equilibrium pricing formulas by applying the arbitrage-free pricing principle. Subsequently, we examine the distinctions between various contracts.

I. "Arbitrage-Free" Pricing Principle

II. Forward/Futures Pricing

   A. Stocks without Dividends

   B. Stocks with Discrete Fixed Dividends

   C. Stocks with Continuous Dividend Yield

   D. Foreign Currencies (FX)

III. Summary of Pricing Formulas

   A. Financial Forwards and Futures

   B. Value of the Forward Contract

Arbitrage-Free Pricing

Here, we introduce one of the most important principles in all of finance, i.e., arbitrage-free pricing.

- We start by identifying and pricing a portfolio replicating the payoffs produced by the forward/futures contract.
- If the value of the replicating portfolio differs from the value of the forward contract, this represents an arbitrage opportunity; investors can implement a trading strategy that does not require a cash input but has some positive probability of making profits without risking a loss.
- If the value of the replicating portfolio is less (greater) than the value of the forward, then one can earn positive profits with zero risk and zero net investment by selling (buying) forward, buying (shorting) the underlying, and borrowing (lending) money. "Arbitrage-free" pricing implies that such profits vanish quickly.

The Law of One Price

- Arbitrage-free pricing is based on the "law of one price," which dictates that assets with the same payoffs must have the same price; otherwise, profitable arbitrage trading opportunities exist.
- We'll employ the law of one price throughout the semester to determine arbitrage-free prices for various financial derivatives.
- We'll look for a portfolio of assets that has exactly the same payoffs as the derivative of interest.
- We'll price the derivative by pricing the replicating portfolio and invoking the law of one price.
- Note that the replicating portfolio is a "synthetic" derivative.
- The procedure for identifying and pricing replicating portfolios is most intuitive with forwards/futures, which is why we start with these.

Pricing Forward and Futures

Intuition: The forward price should be the price of "holding the spot" until maturity; i.e., $F(t,T) = S(t)e^{r(T-t)}$.

- If the price exceeds the cost of holding the spot until delivery, no one will buy forward since it is cheaper to buy the underlying today.
- Understand this, and you'll understand forwards!

An Example: Stocks without Dividends

- Suppose you hold 10,000 shares of stock priced at \$100/share.
- You can either sell today or commit to selling in six months.
- The risk-free interest rate is 4%, and the term structure is flat.
- What is the 6-month forward/futures price of the stock? $F(t,T) = 100e^{0.04(.5)} = 102.02$.

Pricing Forwards/Futures: No-Arbitrage Argument

The forward/futures price must equal \$102.02 /share.

- What if the price is \$103.00 /share?

Transaction	**Payoff now**	**Payoff @ T**
Sell Forward	\$0	\$103-S_T
Buy Stock	(\$100)	S_T
Borrow	\$100	(\$100)$e^{.04(.5)}$ = (\$102.02)
Net Profit	\$0	\$0.98

- What if the price is \$101.00/ share?

Transaction	**Payoff now**	**Payoff @ *T***
Buy Forward	\$0	\$$S_T$ - 101
Short Stock	\$100	-S_T
Lend	(\$100)	\$100$e^{.04(.5)}$ = \$102.02
Net Profit	\$0	\$1.02

Therefore, if $F(t,T) \neq S(t)e^{r(T-t)}$, then there is an arbitrage opportunity!

Stocks with Discrete Fixed Dividends

- Most stocks pay dividends. Assume dividends are known to be paid at specific points in time.
- Next, we recycle the previous "Stocks without Dividends" numerical example; i.e., the spot price of the stock is \$100 per share and the annual rate of interest is 4%. However, we assume that the stock will pay a \$1 dividend 3 months from now. What is the 6-month forward/futures price?
- Intuition: The forward price must be the price of "holding the spot (net of the present value of dividends) until maturity; i.e.,

$$
\begin{aligned}
F(t,T) &= [S(t) - PV(D)]e^{r(T-t)} \\
&= [100 - .99)]e^{0.04(.5)} = \$101.01.
\end{aligned}
$$

Therefore, the forward/futures price must equal \$101.01/share.

- What if the price is \$102.00 /share?

Transaction	**Payoff now**	**Payoff @ t (D)**	**Payoff @ T**
Sell Forward	\$0	\$0	\$102 - S_T
Buy Stock	(\$100)	\$1	S_T
Borrow	\$100	(\$1)	-\$100$e^{.04(.25)}$ =-\$101.01
Net Profit	\$0	\$0	\$0.99

- What if the price is \$100.00/ share?

Transaction	**Payoff now**	**Payoff @ t (D)**	**Payoff @ T**
Buy Forward	\$0	\$0	S_T - \$100
Sell Stock	\$100	(\$1)	- S_T
Lend	(\$100)	\$1	\$100$e^{.04(.25)}$ = \$101.01
Net Profit	\$0	\$0	\$1.01

Therefore, if $F(t,T) \neq [S(t) - PV(D)]e^{r(T-t)}$, then there is an arbitrage opportunity.

Stocks with Continuous Dividend Yield

Assume dividends, or costs, are proportional to the security price and are paid continuously.

Example: 6-month forward/futures contract on a stock index.

Suppose $S(t)$ is \$900, the interest rate r is 4%, and the dividend yield δ is 3%. What is the price of a 6-month forward/futures contract on the stock market index?

Equilibrium Argument

The "underlying" is not 1 share of the index. If you buy 1 share of the index, hold it, and reinvest the dividends proportionally in the index, you will have $e^{\delta(T-t)} = e^{03(.5)} = 1.015111$ shares of the index at maturity.

Consequently, the "underlying" consists of $e^{-\delta(T-t)}$ shares because at maturity $e^{-\delta(T-t)}$ shares yield $e^{-\delta(T-t)} \times e^{\delta(T-t)} = 1$ share of the index. The cost of the underlying security is:

$$V(t) = S(t)e^{-\delta(T-t)} = 900 \times e^{-03(.5)} = \$886.60.$$

Therefore, the index forward/futures price is:

$$F(t,T) = V(t)e^{r(T-t)} = S(t)e^{-\delta(T-t)}e^{r(T-t)} = Se^{(r-\delta)(T-t)}$$
$$= 886.60e^{0.04(.5)} = \$904.51.$$

No-Arbitrage Argument

- The forward/futures price must equal \$904.51. What if the price is \$905.00 ?

Transaction	**Payoff now**	**Payoff @ T**
Sell Forward	\$0	\$905 - S_T
Buy Index	(\$886.60)	S_T
Borrow	\$886.60	$\$886.60e^{.04(.5)} = \$904.51$
Net Profit	\$0	\$0.49

- What if the price is \$903.00 ?

Transaction	Payoff now	Payoff @ T
Buy Forward	\$0	S_T - \$903
Sell Index	\$886.60	$-S_T$
Lend	\$886.60	$\$886.60e^{.04(.5)} = \$904.51$
Net Profit	\$0	\$1.51

Therefore, if $F(t,T) \neq Se^{(r-\delta)(T-t)}$, then there is an arbitrage opportunity.

Foreign Currencies (FX)

- Foreign currency forwards/futures resemble index forwards/futures.
- A unit of foreign currency can be viewed as a stock that pays a continuous dividend yield equal to the foreign interest rate.

Covered Interest Rate Parity:

- After adjusting for exchange rate differentials, interest rates must be equal across countries.

Example:

- Suppose $r_{us} = 7.41\%, r_{\text{Swiss}} = 8.87\%$, and the USD/SF spot exchange rate $S(t)$ is \$0.6667; i.e., one Swiss Franc is currently worth 2/3 of one dollar. What is the price of a 4-month USD/SF forward/futures contract?

- The "underlying asset" here is not worth 1 Swiss Franc. Instead, it is worth $V(t) = S(t) \times e^{-r_{\text{Swiss}}(T-t)}$; thus,

$$F(t,T) = V(t)e^{r(T-t)} = S(t)e^{-r_{\text{Swiss}}(T-t)}e^{r_{us}(T-t)}$$
$$= S(t)e^{(r_{us}-r_{\text{Swiss}})(T-t)} = .6667e^{(0.074-0.087//3} = .6634.$$

- What if the price is \$0.65 ?

Transaction	Payoff now	Payoff @ *T*
Buy Forward	\$0	S_T - \$.65
Sell USD/SF	\$0.6634	$-S_T$
Lend	(\$0.6634)	$\$.6634e^{.0741/3} = \$.68$
Net Profit	\$0	\$0.03

- What if the price is \$0.70 ?

Transaction	Payoff now	Payoff @ *T*
Sell Forward	\$0	\$.70 - S_T
Buy USD/SF	(\$0.6634)	S_T
Borrow	\$0.6634	$\$.6634e^{.0741/3} = \$.68$
Net Profit	\$0	\$0.02

If $F(t,T) \neq S(t)e^{(r_{us}-r_{\text{Swiss}})(T-t)}$, there is an arbitrage opportunity.

Summary of Pricing Formulas: Financial Forwards and Futures

- Arbitrage-free prices are given by:

$$F(t,T) = V(t) \times \mathrm{e}^{r(t,T)\times(T-t)}$$

where

$F(t,T)$ = Forward/futures price.

$r(t,T)$ = Riskless interest rate between now and later (t and T).

$V(t)$ = Value of the "right" underlying

The trick is always to determine $V(t)$! The "right underlying" or strategies $V(t)$ include:

- Stock without dividends

Buy one share at time t and hold until time T :

$$V(t) = S(t)$$

- Stock with known dividends

Buy one share at time t, borrow the present value of the dividends, pay off the debt with the dividends, and hold until time T:

$$V(t) = S(t) - \mathrm{PV(D)}$$

- Stock with known dividend yield δ

Buy $\mathrm{e}^{-\delta\times(T-t)}$ shares at time t, reinvest the dividends in stock, and hold until time T :

$$V(t) = S(t) \times \mathrm{e}^{-\delta\times(T-t)}$$

- Foreign currency (FX)

Buy $\mathrm{e}^{-r_{\text{foreign}}\times(T-t)}$ units of the currency at time t, earn foreign interest, and hold until time T :

$$V(t) = S(t) \times \mathrm{e}^{-r_{\text{foreign}}\times(T-t)}$$

Value of Forward Contract

The "forward price" is not the value of the forward contract! What is the value of a forward contract?

- To find the value of a forward contract after initiation ($f(t)$), compare the following:
  - Long one forward contract worth $f(t)$ at time t.
  - Long one unit of the underlying security worth $S(t)$ at time t and $K \times \mathrm{e}^{-r\times(T-t)}$ of borrowing.

Both positions pay off $S(T) - K$ at time T, and must have the same value at time t. Therefore, the value of a forward contract at time t corresponds to the value of the replicating portfolio at time t, which is

$$f(t) = S(t) - K \times \mathrm{e}^{-r\times(T-t)}$$