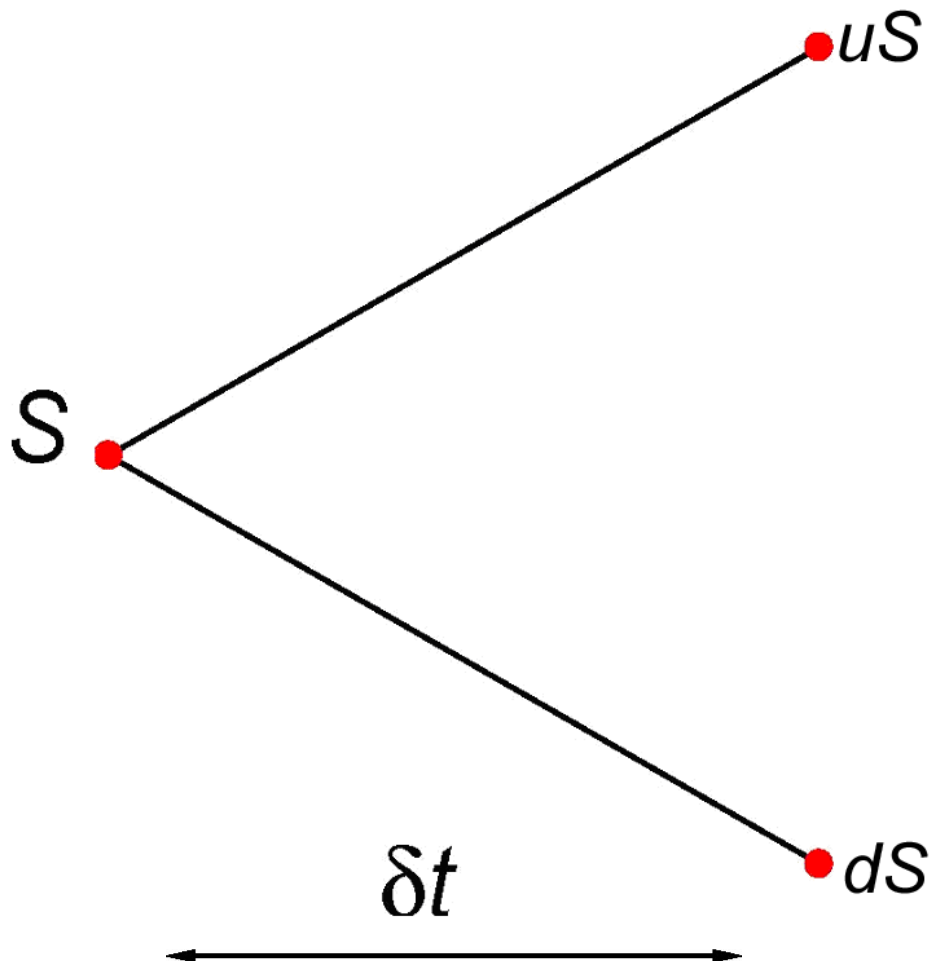
Binomial Trees

Today's Agenda

- 3 Approaches to pricing options
  - Replicating Portfolio Approach
  - Delta Hedging Approach
  - Risk Neutral Valuation Approach
- Multiperiod option pricing

Binomial Model: One Timestep



Replicating Portfolio Approach

- **Example:**
  - $u = 1.05$
  - $d = 0.95$
  - $p = 0.60$
  - Current asset price S is 100
- i.e., there is a 60% chance that in one month, the stock price will be \$105 and a 40% chance it will be \$95.

Replicating Portfolio Approach (Calls)

- Suppose we form a portfolio today, comprising Δ shares and B dollars in riskless bonds, which replicates call option payoffs after one timestep.
- The initial cost of this portfolio is $V_{RP} = \Delta S + B$; since it replicates call option payoffs after one timestep, the value of this "replicating" portfolio must be the same as the value of the call option; i.e., $V_{RP} = c$.
- After one timestep, the value of the replicating portfolio at the up node is $V_{RP}^{u} = \Delta uS + e^{r\delta t}B = c_u = \max[0, uS - K]$, and at the down node, it is worth $V_{RP}^{d} = \Delta dS + e^{r\delta t}B = c_d = \max[0, dS - K]$.

Replicating Portfolio Approach (Calls)

- Thus, we have two equations (for c_u and c_d) in two unknowns (Δ and B).
- Subtracting the c_d equation from the c_u equation and solving for Δ, we find that $\Delta = \dfrac{c_u - c_d}{S(u-d)} \geq 0.$
- Substituting $\Delta = \dfrac{c_u - c_d}{S(u-d)}$ into either the c_u or c_d equation, we find that $B = \dfrac{uc_d - dc_u}{e^{r\delta t}(u-d)} \leq 0.$

Replicating a Call Option

Recall our previous numerical example. If u =1.05 and d =0.95, then the date δt values for S_u, S_d, c_u, and c_d are \$105, \$95, \$5, and \$0 respectively:

	$S_u = \$105$
	$c_u = \$5$
$S = \$100$	
	$S_d = \$95$
	$c_d = \$0$

Replicating a Call Option

Let's input the values from our previous numerical example:

$$\Delta = \frac{c_u - c_d}{uS - dS} = \frac{5}{10} = 0.5, \text{ and}$$

$$B = \frac{uc_d - dc_u}{e^{r\delta t}(u-d)} = \frac{1.05(0) - .95(5)}{1.0042(0.10)} = -47.30;$$

$$\therefore c = \Delta S + B = 0.50(100) - 47.30 = \$2.70.$$

Replicating Portfolio Approach (Puts)

- Suppose we form a portfolio today, comprising Δ shares and B dollars in riskless bonds, which replicates put option payoffs after one timestep.
- The initial cost of this portfolio is $V_{RP} = \Delta S + B$; since it replicates put option payoffs after one timestep, the value of this "replicating" portfolio must be the same as the value of the put option; i.e., $V_{RP} = p$.
- After one timestep, the value of the replicating portfolio at the up node is $V_{RP}^{u} = \Delta uS + e^{r\delta t}B = p_u = \max[0, K - uS]$, and at the down node, it is worth $V_{RP}^{d} = \Delta dS + e^{r\delta t}B = p_d = \max[0, K - dS]$.

Replicating Portfolio Approach (Puts)

- Thus, we have two equations (for p_u and p_d) in two unknowns (Δ and B).
- Subtracting the p_d equation from the p_u equation and solving for Δ, we find that $\Delta = \dfrac{p_u - p_d}{S(u-d)} \leq 0.$
- Substituting $\Delta = \dfrac{p_u - p_d}{S(u-d)}$ into either the p_u or p_d equation, we find that $B = \dfrac{up_d - dp_u}{e^{r\delta t}(u-d)} \geq 0.$

Replicating a Put Option

Recall our previous numerical example. If u =1.05 and d =0.95, then the date δt values for S_u, S_d, p_u, and p_d are \$105, \$95, \$0, and \$5 respectively:

	$S_u = \$105$
	$p_u = \$0$
$S = \$100$	
	$S_d = \$95$
	$p_d = \$5$

Replicating a Put Option

Let's input the values from our previous numerical example:

$$\Delta = \frac{p_u - p_d}{uS - dS} = \frac{-5}{10} = -0.5, \text{ and}$$

$$B = \frac{up_d - dp_u}{i(u-d)} = \frac{1.05(5) - .95(0)}{1.0042(0.10)} = 52.28;$$

$$\therefore\ p = \Delta S + B = -0.50(100) + 52.28 = \$2.28.$$

Delta Hedging Approach (Calls)

- Suppose we hold a call option on this asset with an exercise price of 100 that will expire in 1 month ($\delta t = 1$ month).
- Holding just the stock or the option is risky:
  - Holding just the stock: If the stock rises we have \$105, a profit of \$5. If it falls we have \$95, a loss of \$5.
  - Holding just the option: If the stock rises we get a payoff of \$5. If it falls, the option expires out of the money.

Delta Hedging Approach (Calls)

- Let's sell short a quantity Δ of the underlying asset so that now we have a portfolio consisting of a long call option position and Δ short stock position.
- If the asset rises to 105 we have a portfolio worth

$$Max(105 - 100, 0) - \Delta \times 105 = 5 - 105\Delta.$$

- If the asset falls we have

$$Max(95 - 100, 0) - \Delta \times 95 = -95\Delta.$$

Delta Hedging Approach (Calls)

- Suppose we choose Δ such that $5 - 105\Delta = -95\Delta$, i.e., $\Delta = ½$.
- Then the payoff on our portfolio one month from now is $-\$47.50$, irrespective of whether the stock goes up or down!
- Note that since our portfolio has a riskless payoff, then it must earn the riskless rate of return over the next month.

Delta Hedging Approach (Calls)

- If C is the option value today then our portfolio's value is currently $C - \Delta \times 100$.
- The present value of the payoff on this portfolio is $-e^{-rt}(\$47.50)$.
- Suppose $r = 5\%$; then

$$C - \Delta \times 100 = -e^{-rt}(\$47.50)$$

$$\Rightarrow C = 50 - e^{-(.05)(1/12)}(\$47.50)$$

$$C = 50 - 47.30 = \$2.70.$$

Delta Hedging Approach (Puts)

- Let's purchase a quantity Δ of the underlying asset so that now we have a portfolio consisting of a long put option position and Δ long stock position.
- If the asset rises to 105 we have a portfolio worth

$$Max(100 - 105, 0) + \Delta \times 105 = 105\Delta.$$

- If the asset falls we have

$$Max(100 - 95, 0) + \Delta \times 95 = 5 + 95\Delta.$$

Delta Hedging Approach (Puts)

- Suppose we choose Δ such that $105\Delta = 5 + 95\Delta$, i.e., $\Delta = ½$.
- Then the payoff on our portfolio one month from now is $52.50, irrespective of whether the stock goes up or down!
- Note that since our portfolio has a riskless payoff, then it must earn the riskless rate of return over the next month.

Delta Hedging Approach (Puts)

- If P is the option value today then our portfolio's value is currently $P + \Delta \times 100$.
- The present value of the payoff on this portfolio is $e^{-rt}(\$52.50)$.
- Suppose $r = 5\%$; then

$$P + \Delta \times 100 = e^{-rt}(\$52.50)$$

$$\Rightarrow P = e^{-rt}(\$52.50) - \Delta \times 100$$

$$P = 52.28 - 50 = \$2.28.$$

Risk Neutral Valuation Approach

- Define μ as the annualized expected rate of return on the stock. Note that:
  - $E(S_{\delta t}) = puS + (1\text{-}p)dS = e^{\mu\delta t}S;$

$\therefore\ pu + (1\text{-}p)d = e^{\mu\delta t}$, and $p = (e^{\mu\delta t} - d)/(u - d)$.

- Since $p = .60$, $\mu = \dfrac{\ln\left(pu + (1-p)d\right)}{\delta t} =$

$$\frac{\ln\left(.6(1.05) + (.4).95\right)}{.0833} = 11.94\%$$

Risk Neutral Valuation Approach

- Suppose $r = 5\%$. Then the *risk neutral probability* (q) is $q = (e^{r\delta t} - d)/(u\text{-}d) = .5418.$
- The value of the call option is:

$$C_0 = e^{-r\delta t}\begin{pmatrix} qMax(uS-K,0) \\ +(1-q)Max(dS-K,0) \end{pmatrix}$$

$$= .9958(.5418(5)) = \$2.70.$$

Risk Neutral Valuation Approach

- Using risk neutral valuation, the value of an otherwise identical put option is:

$$P_0 = e^{-r\delta t}\begin{pmatrix} qMax(K-uS,0) \\ +(1-q)Max(K-dS,0) \end{pmatrix}$$

$$= .9958(.4582(5)) = \$2.28.$$

Binomial Tree for a One-Step Call Option

Risk Neutral Valuation & Replicating Portfolio Approaches to Pricing a Call Option (One Timestep)

Asset	$100
u	1.05
d	0.95
q	0.5418
dt	0.0833
Interest rate	5%
Discount factor	0.9958
Strike	$100

Risk Neutral Valuation	Stock
	Delta
	Option
	Long Stock
Replicating Portfolio	Borrowing
	Net Value

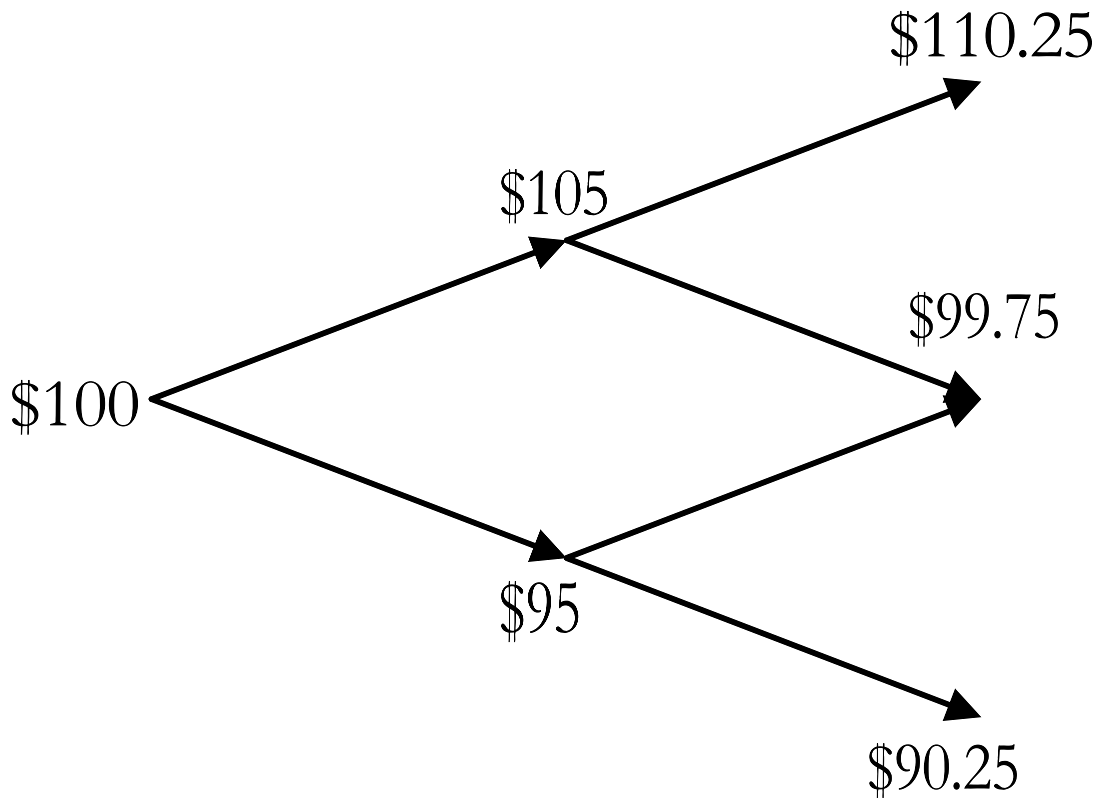
Binomial Tree for a One-Step Put Option

Risk Neutral Valuation & Replicating Portfolio Approaches to Pricing a Put Option (One Timestep)

Asset	$100	**One timestep**	**expiration**
u	1.05	**before expiration**	
d	0.95		
q	0.5418		$105.00
dt	0.0833		
Interest rate	5%	$100.00	$0.00
Discount factor	0.9958	-50.00%	
Strike	$100	$2.28	
		-$50.00	
Risk Neutral Valuation	Stock	$52.28	
	Delta	$2.28	
	Option		$95.00
	Short Stock		
Replicating Portfolio	Lending		$5.00
	Net Value		

Implications of adding a Time Step

- Now suppose that we add another time-step; i.e., time to expiration is now 2 years rather than 1 year. This results in the following binomial tree:



Implications of adding a Time Step

- Since uuS = \$110.25, udS = \$99.75, and ddS = \$90.25, this implies that
  - $c_{uu} = Max[110.25\text{-}100,0]$ =\$10.25 and $p_{uu} = Max[100\text{-}110.25,0]$ = \$0,
  - $c_{ud} = Max[99.75\text{-}100,0]$ = \$0 and $p_{ud} = Max[100\text{-}99.75,0]$ = \$0.25, and
  - $c_{dd} = Max[90.25\text{-}100,0]$ =\$0 and $p_{dd} = Max[100\text{-}90.25,0]$ = \$9.75.
- Since the risk neutral probability of an up move is .5418 and the interest rate is 5%, this implies the following prices for c_u, p_u, c_d, p_d, c, and p:
  - $c_u = e^{-.05/12}[(.5418)10.25 + (.4582)0]$ = \$5.53 and

  $p_u = e^{-.05/12}[(.5418)0 + (.4582)0.25]$ = \$0.11,
  - $c_d = 0$ and $p_d = e^{-.05/12}[(.5418).25 + (.4582)9.75]$ = \$4.58,
  - $c = e^{-.05/12}[(.5418)5.53]$ = \$2.98 and $p = e^{-.05/12}[(.5418).11 + (.4582)4.58]$ = \$2.15.

Binomial Tree for a Two-Step Call Option

Risk Neutral Valuation & Replicating Portfolio Approaches to Pricing a Call Option (Two Timesteps)

Asset	$100
u	1.05
d	0.95
q	0.5418
dt	0.0833
Interest rate	5%
Discount factor	0.9958
Strike	$100

Risk Neutral Valuation	Stock
	Delta
	Option
	Long Stock
Replicating Portfolio	Borrowing
	Net Value

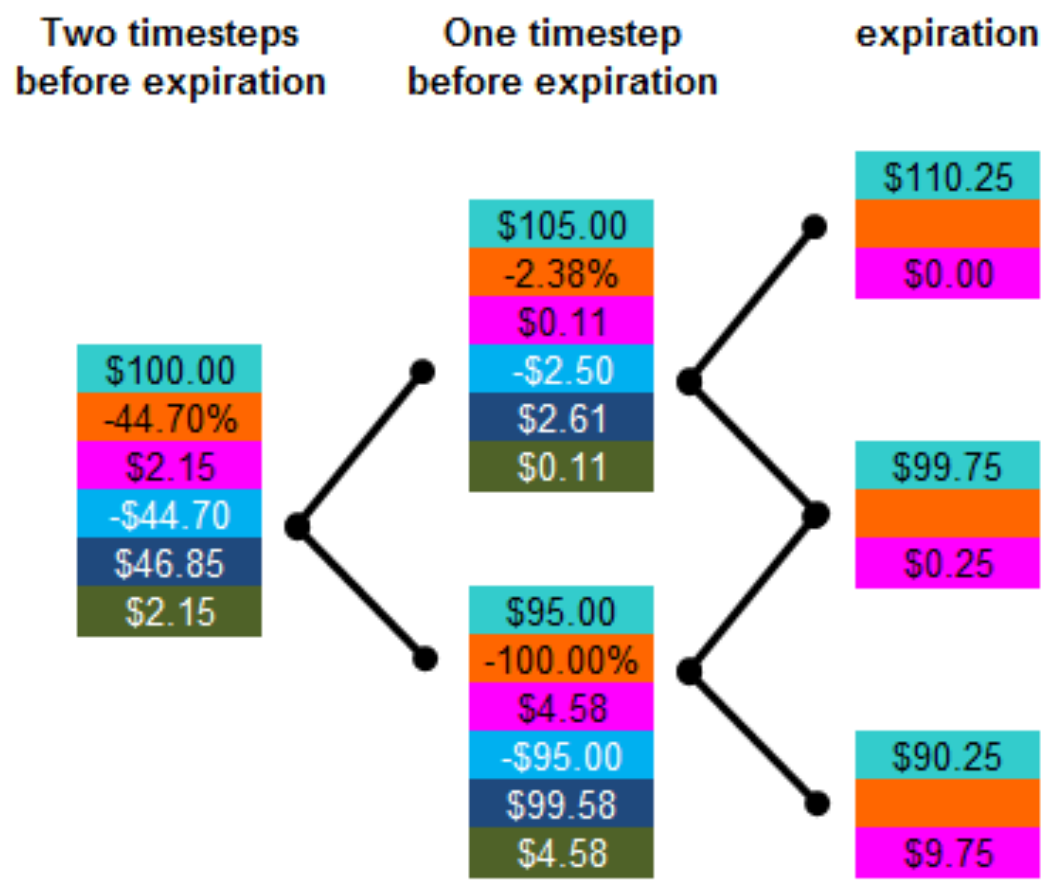
Binomial Tree for a Two-Step Put Option

Risk Neutral Valuation & Replicating Portfolio Approaches to Pricing a Put Option (Two Timesteps)

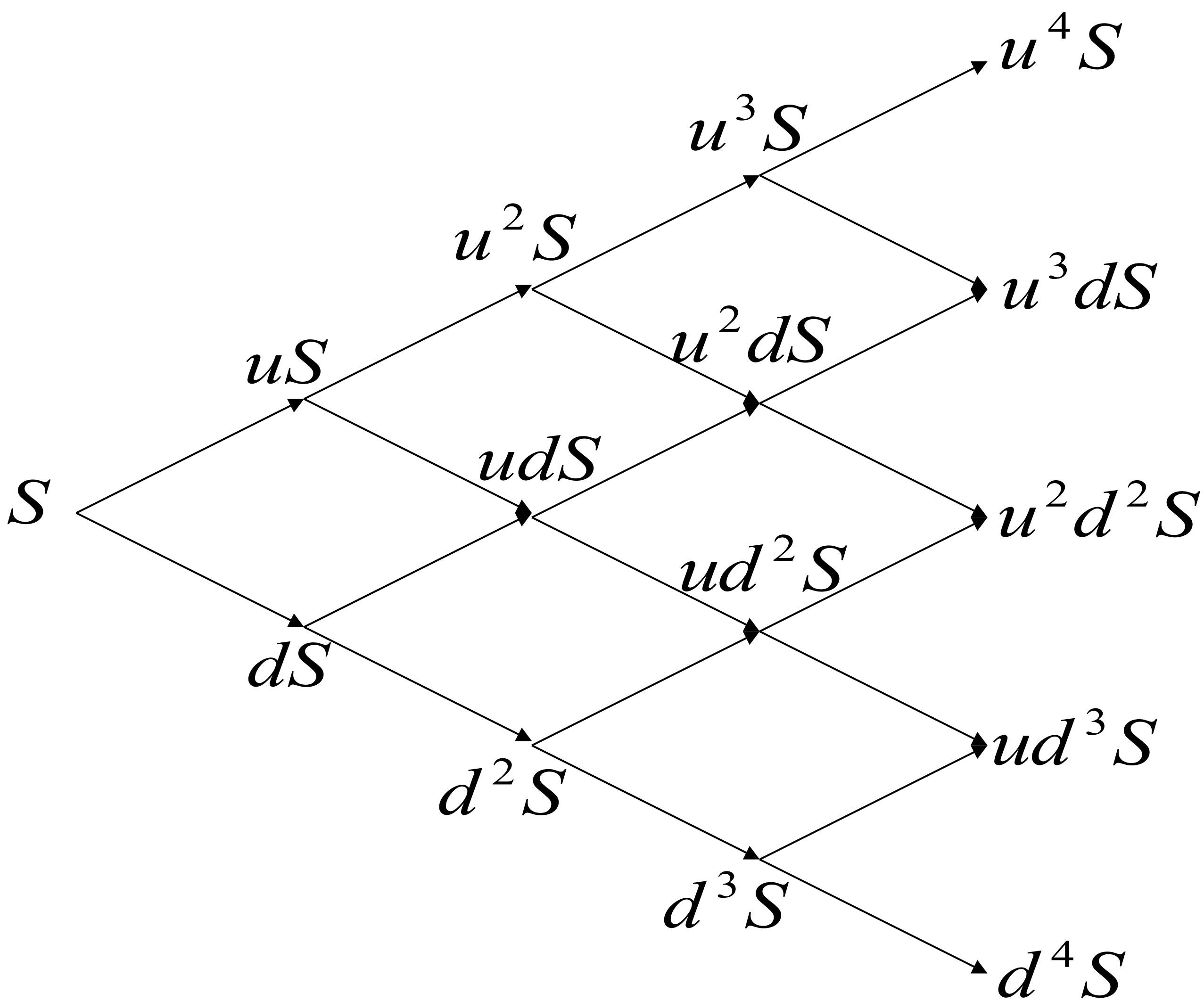
Asset	$100
u	1.05
d	0.95
q	0.5418
dt	0.0833
Interest rate	5%
Discount factor	0.9958
Strike	$100

Risk Neutral Valuation: Stock, Delta, Option

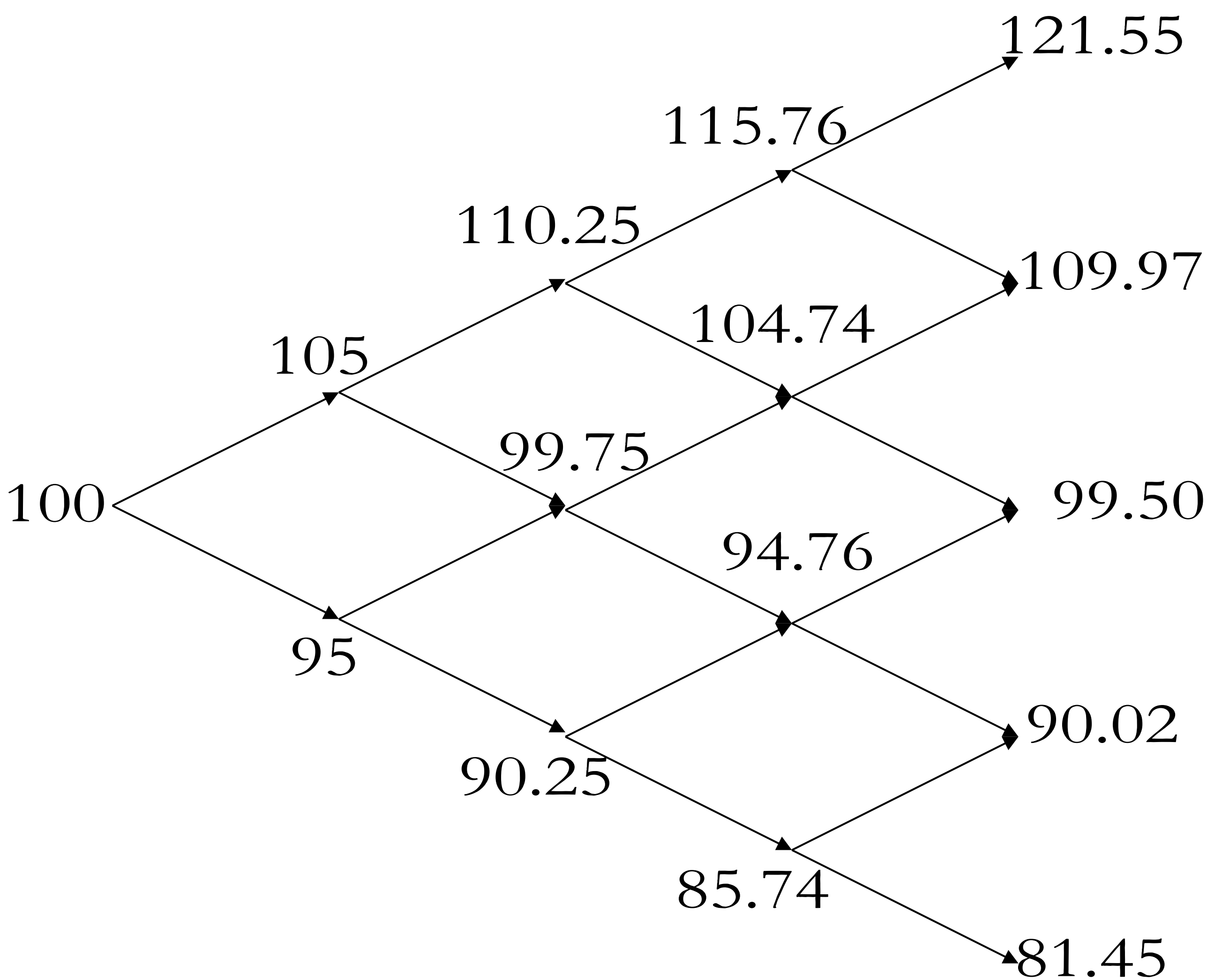
Replicating Portfolio: Short Stock, Lending, Net Value



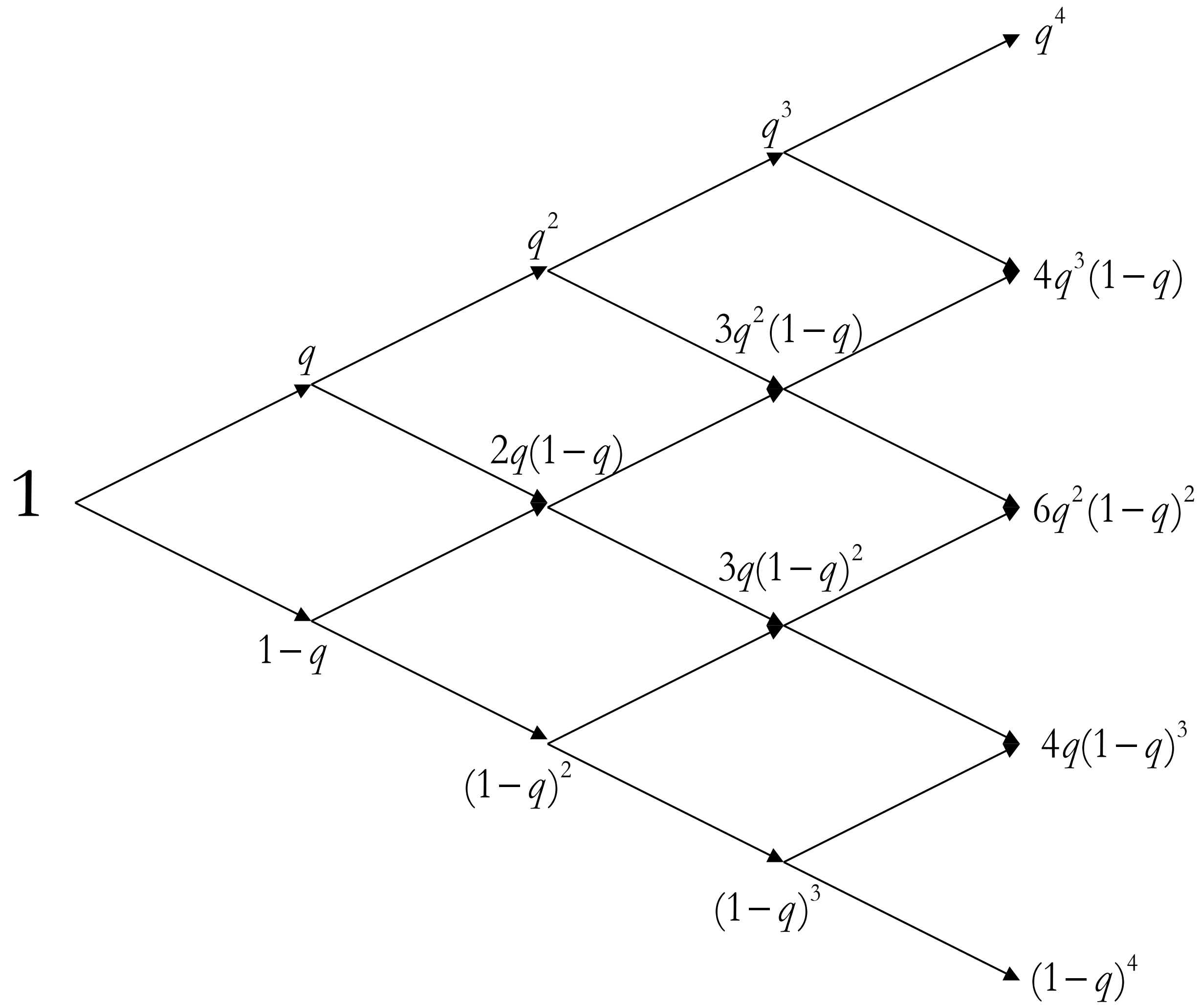
Implications of even more Time Steps



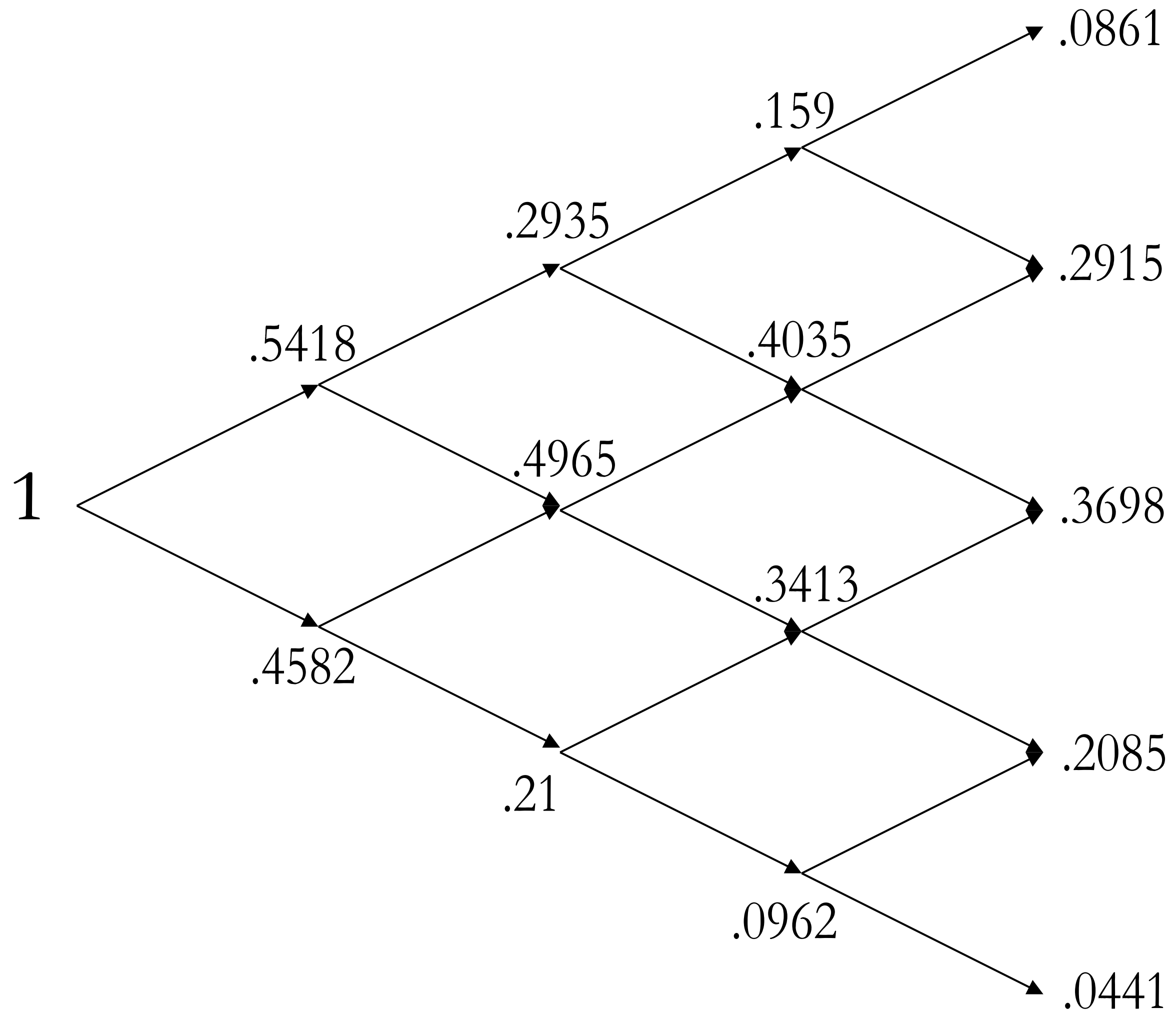
Implications of even more Time Steps



Implications of even more Time Steps



Implications of even more Time Steps



1-4 Time Step Call Option Prices

1. n = 1: By inspection, the call option is only in-the-money at the up (u) node. For one timestep,
$$c = e^{-r\delta t}[qc_u] = 0.9958(0.5418)(5) = \$2.70.$$
2. n = 2: By inspection, the call option is only in-the-money at the up-up ($\underline{u}u$) node. For two timesteps,
$$c = e^{-2r\delta t}[q^2 c_{uu}] = 0.9958^2[.5418^2(10.25)] = \$2.98.$$

3. n = 3: By inspection, the call option is only in-the-money at the up-up-up (uuu) node and the up-up-down (uud) node. For three timesteps,
$$c = e^{-3r\delta t}[q^3 c_{uuu} + 3q^2(1-q)c_{uud}]$$
$$= 0.9958^3\left[(0.5418^3)(15.76) + 3(0.5418^2)(0.4582)(4.74)\right] = \$4.36.$$
4. n = 4: By inspection, the call option is only in-the-money at the up-up-up-up ($\underline{u}uuu$) and up-up-up-down ($\underline{u}uud$) node. For four timesteps,
$$c = e^{-4r\delta t}[q^4 c_{uuuu} + 4q^3(1-q)c_{uuud}]$$
$$= 0.9958^4\left[(0.5418^4)(21.55) + 4(0.5418^3)(0.4582)(9.97)\right] = \$4.68.$$

1-4 Time Step Put Option Prices

Now that we know arbitrage-free prices for 1-4 time step call options, apply the put-call parity equation to determine arbitrage-free prices for 1-4 time step put options. Keep in mind that for n time steps,

$$c + Ke^{-rn\delta t} = p + S.$$

The Cox-Ross-Rubinstein (CRR) call equation

- The complexity of analysis grows with each additional time-step. Fortunately, Cox, Ross, and Rubinstein (CRR) provide a recursive multi-period call option pricing formula:

$$C = e^{-rT} \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} q^j (1-q)^{n-j} C_j.$$

- $\frac{n!}{j!(n-j)!}$ indicates how many path sequences exist for each of the $n+1$ terminal nodes;
- $q^j (1-q)^{n-j}$ corresponds to the risk-neutral probability of one j up and $n-j$ down move path sequence;
- $\frac{n!}{j!(n-j)!} q^j (1-q)^{n-j}$ indicates the risk-neutral probability of the stock price ending up at the $j, n-j$ terminal node;
- $C_j = Max\left[0, u^j d^{n-j} S - K\right]$; and
- $T = n\delta t$ corresponds to a fixed expiration date T periods from now.

The Cox-Ross-Rubinstein (CRR) call equation

- Since $C_j = Max\left(0, u^j d^{n-j} S - K\right)$, we need to determine the minimum number of up moves such that the call option expires in-the-money; i.e., so that $u^j d^{n-j} S > K$.
- Let b represent the *non-integer* value for j such that $u^b d^{n-b} S = K$. Solving this equation for b,

$$\ln\left(u^b d^{n-b} S\right) = \ln K$$
$$b \ln u + (n-b)\ln d = \ln(K/S);$$
$$b \ln(u/d) = \ln\left(K/Sd^n\right);$$
$$b = \ln\left(K/Sd^n\right)/\ln(u/d).$$

- The minimum *integer* value for j is a, obtained by rounding to the nearest integer greater than b.
- If $a = 0$, then the call is in-the-money at *all* $n+1$ terminal nodes.
- If $a = n+1$, the call is out-of-the-money at *all* $n+1$ terminal nodes.

The Cox-Ross-Rubinstein (CRR) call equation

- Having determined the *minimum* number of up moves (a) required for $C_j > 0$, if follows that $C_j > 0$ for $j = a, \ldots, n$. Then the risk neutral valuation formula for pricing such an option is:

$$C = SB_1 - Ke^{-rT}B_2,$$

where

$$B_1 = \sum_{j=a}^{n} \left(\frac{n!}{j!(n-j)!} \right) \cdot q^j \cdot (1-q)^{n-j} \cdot \left(u^j \cdot d^{n-j} \cdot e^{-rn\delta t} \right);\text{ and}$$

$$B_2 = \sum_{j=a}^{n} \left(\frac{n!}{j!(n-j)!} \right) \cdot q^j \cdot (1-q)^{n-j}.$$

The Black-Scholes-Merton (BSM) call equation

- As in the previous slide, suppose the time to expiration $T = n\delta t$. Now consider the "limiting" case where $n \to \infty$ and $\delta t \to 0$ for a fixed value of T. When this occurs, the binomial risk neutral probabilities B_1 and B_2 that appear in the *CRR* option pricing formula converge in probability to the standard normal probabilities $N(d_1)$ and $N(d_2)$, where

$$d_1 = \frac{\ln(S/K) + (r + .5\sigma^2)T}{\sigma\sqrt{T}} \text{ and } d_2 = d_1 - \sigma\sqrt{T}.$$

- Then the risk neutral valuation formula for pricing such an option is:

$$C = SN(d_1) - Ke^{-rT}N(d_2).$$

This formula was independently published by Black and Scholes and by Merton in 1973, so it is commonly referred to as the BSM call option pricing formula. Scholes and Merton were awarded the Nobel Prize for Economics in 1997 for this discovery; Black was not named since he passed away in 1995 and the Nobel Prize is not posthumously given.

The CRR and BSM put equations

- The CRR and BSM put equations are obtained by invoking the put-call parity theorem. Since the only difference between these equations is that CRR is based upon the standard binomial distribution function (as captured by B_1 and B_2) whereas BSM is based upon the normal distribution function (as captured by $N(d_1)$ and $N(d_2)$), the CRR and BSM put equations are otherwise identical to each other.
- According to the put call parity theorem, the BSM put equation is

$$\begin{aligned} p &= c + Ke^{-rT} - S \\ &= SN(d_1) - Ke^{-rT}N(d_2) + Ke^{-rT} - S \\ &= Ke^{-rT}(1 - N(d_2)) - S(1 - N(d_1)). \end{aligned}$$

- By symmetry, it follows that the CRR put equation is

$$p = Ke^{-rT}(1 - B_2) - S(1 - B_1).$$

CRR & BSM call and put prices – numerical example

Suppose $S = \$100$, $K = \$100$, $\sigma = .20$, $n = 2$, $\delta t = .25$, $T = n\delta t = 2(.25) = .5$, and $r = .03$. Also suppose that $u = e^{\sigma\sqrt{\delta t}} = e^{.2\sqrt{.25}} = 1.1052$, $d = 1/u = .9048$, and $q = \dfrac{e^{r\delta t} - d}{u - d} = .5126$. What are the CRR call and put prices, given these parameters?

SOLUTION: Here's the two-timestep stock tree:

		\$122.14
	\$110.52	
\$100.00		\$100.00
	\$90.48	
		\$81.87

Therefore, the only node at which this call is in-the-money is a node uu; specifically, $c_{uu} = \max[0, S_{uu} - K] = \$22.14$ and $c_{ud} = c_{dd} = 0$. Then

$$c = e^{-rT}[q^2 c_{uu}] = .9851(.5126^2 \cdot \$22.14) = \$5.73, \text{ and}$$

$$p = c + e^{-rT}K - S = \$5.73 + \$98.51 - \$100 = \$4.24.$$

CRR & BSM call and put prices – numerical example

What are the BSM call and put prices, given the parameters from the preceding page?

SOLUTION: First calculate the standard normal probabilities $N(d_1)$ and $N(d_2)$.

$$d_1 = \frac{\ln(S/K) + (r + .5\sigma^2)T}{\sigma\sqrt{T}} = \frac{\ln(100/100) + (.03 + .5(.04)).5}{.2\sqrt{.5}} = .1768, \text{ and}$$

$$d_2 = d_1 - \sigma\sqrt{T} = .1768 - .2\sqrt{.5} = .0354.$$

Thus, $N(d_1) = .5702$ and $N(d_2) = .5141$, and

$$c = SN(d_1) - Ke^{-rT}N(d_2) = 100(.5702) - 100e^{-.03(.5)}(.5141) = \$6.37, \text{ and}$$

$$p = Ke^{-rT}(1 - N(d_2)) - S(1 - N(d_1)) = 100e^{-.03(.5)}(.4859) - 100(.4298) = \$4.88.$$

CRR & BSM call and put prices – numerical example

- When there are only two timesteps until expiration 6 months from now, the CRR model produces call and put prices of \$5.73 and \$4.24, compared with BSM model prices of \$6.37 and \$4.88.
- However since the standard binomial distribution functions (B_1 and B_2) converge in probability to the standard normal distribution functions ($N(d_1)$ and $N(d_2)$), CRR and BSM prices also converge rather quickly. Here's a table illustrating this for 1 through 5,000 timesteps occurring during the course of a 6-month time to expiration:

Timesteps	BSM Call	BSM Put	CRR Call	CRR Put	Call Difference	Call % Diff	Put Difference
1	\$6.3710	\$4.8822	\$7.7512	\$6.2624	-\$1.3802	-21.66%	-\$1.3802
2	\$6.3710	\$4.8822	\$5.7309	\$4.2421	\$0.6401	10.05%	\$0.6401
5	\$6.3710	\$4.8822	\$6.6501	\$5.1613	-\$0.2791	-4.38%	-\$0.2791
10	\$6.3710	\$4.8822	\$6.2323	\$4.7435	\$0.1387	2.18%	\$0.1387
25	\$6.3710	\$4.8822	\$6.4261	\$4.9373	-\$0.0551	-0.86%	-\$0.0551
50	\$6.3710	\$4.8822	\$6.3430	\$4.8542	\$0.0280	0.44%	\$0.0280
100	\$6.3710	\$4.8822	\$6.3570	\$4.8682	\$0.0140	0.22%	\$0.0140
200	\$6.3710	\$4.8822	\$6.3640	\$4.8752	\$0.0070	0.11%	\$0.0070
400	\$6.3710	\$4.8822	\$6.3675	\$4.8787	\$0.0035	0.05%	\$0.0035
5000	\$6.3710	\$4.8822	\$6.3707	\$4.8819	\$0.0003	0.00%	\$0.0003

Black-Scholes-Merton (BSM) is a "limiting" case of CRR!

- See Cox-Ross-Rubinstein compared with Black-Scholes-Merton spreadsheet!