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Options, Futures, & Other Derivatives Name: SOLUTIONS
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Problem Set 6

Problem 1: Use the following parameter values to price a call and put option by applying the Delta Hedging, Replicating Portfolio, and Risk Neutral Valuation approaches:

- S = current price of (non-dividend paying) underlying asset = \$50;
- K = exercise price = \$50;
- r = annualized riskless rate of interest = 3%;
- σ = annualized volatility (standard deviation of return) for the underlying asset = .1;
- δt = length of time-step in years = 1;
- u = one plus the rate of return on the underlying asset after one up move = $e^{\sigma\sqrt{\delta t}} = e^{.1\sqrt{1}} = 1.1052$; and
- d = one plus the rate of return on the underlying asset after one down move = $e^{-\sigma\sqrt{\delta t}} = e^{-.1\sqrt{1}} = .9048$.

Delta Hedging Approach

Pricing the Call Option

We start by creating a risk-free bond by forming a perfectly hedged portfolio consisting of a long position in a call option and a short position in Δ shares of the underlying asset. At node u, the value of the hedge portfolio is equal to $V_H^u = C_u - \Delta uS$, and at node d, the value of the hedge portfolio is equal to $V_H^d = C_d - \Delta dS$. Since $uS = 50(1.1052) = \$55.26$ and $dS = 50(0.9048) = 45.24$, it follows that $V_H^u = 5.26 - \Delta 55.26$ and $V_H^d = 0 - \Delta 45.24$. Suppose that we select Δ such that the hedge portfolio is riskless; i.e., $V_H^u = V_H^d$. Solving for Δ, we obtain:

$$V_H^u = V_H^d \Rightarrow 50(1.1052) = \$55.26 = -\Delta 45.24 \Rightarrow \Delta = .525.$$

Substituting $\Delta = .525$ into our expressions for V_H^u and V_H^d, we obtain $V_H^u = V_H^d = -\$23.75$. Thus, the value of a riskless hedge portfolio consisting of one call option and a short position in .525 shares of stock is equivalent in value to a short position in a riskless bond; i.e., it is as if we have created a riskless loan in which we borrow a principal value of \$23.05 up front and pay back principal (\$23.05) plus interest (\$0.75) in one year. To avoid arbitrage, the current value of the hedge portfolio, $V_H = C - \Delta 50 = C - .525(50) = C - 26.25$, must be equal to the present value of our short-term position, which is $e^{-r\delta t} V_H^d = -e^{-.03}(23.75) = -\$23.05$; consequently, $C = \$3.20$.

Pricing the Put Option

We can also determine the arbitrage-free price for the put option using the delta hedging approach. Since the prices of a put option and its underlying stock are inversely related, we form a hedge portfolio consisting of a long position in one put option and a long position in Δ shares of stock. [2] The current value of this portfolio is

$$V_H = P + \Delta S = P + \Delta 50.$$

At node u, the value of the hedge portfolio is equal to $V_H^u = P_u + \Delta uS$, and at node d, the value of the hedge portfolio is equal to $V_H^d = P_d + \Delta dS$. Since $uS = \$55.26$ and $dS = \$45.24$, it follows that $V_H^u = 0 + \Delta 55.26$ and $V_H^d = 4.76 + \Delta 45.24$. Suppose we select Δ such that the hedge portfolio is riskless; i.e., $V_H^u = V_H^d$. Solving for Δ, we obtain:

$$V_H^u = V_H^d \Rightarrow \Delta 55.26 = 4.76 + \Delta 45.24 \Rightarrow \Delta = .475$$

Substituting $\Delta = .475$ into our expressions for V_H^u and V_H^d, we obtain $V_H^u = V_H^d = \$26.25$.

Thus, the value of a riskless hedge portfolio consisting of one put option and .475 shares of stock is equivalent in value to a long position in a riskless bond. In order to prevent arbitrage, the current value of this long bond position, $V_H = P + \Delta 50 = \$e^{-.03} 26.25 = \$25.47$. In order to prevent arbitrage, the current value of the hedge portfolio, $V_H = P + \Delta 50 = P + .475(50) = P + 23.75 \Rightarrow P + \$23.75 = \$25.47 \Rightarrow P = \$1.72$.

Replicating Portfolio Approach

Pricing the Call Option

Next, we consider the replicating portfolio approach for determining the arbitrage-free prices of the call and put options. The current value of the replicating portfolio for the call option is $V_{RP}^C = \Delta_C S + B_C$, where $\Delta_C = \dfrac{C_u - C_d}{S(u-d)} = \dfrac{5.26}{50(.2003)} = .525$ and $B_C = \dfrac{uC_d - dC_u}{e^{r\delta t}(u-d)} = \dfrac{-.9048(5.26)}{e^{.03}(.2003)} = -\$23.05$. Therefore, we can replicate the call option purchasing 0.525 of a share for \$26.25 and borrowing \$23.05, which implies that $C = V_{RP}^C = 26.25 - 23.05 = \$3.20$.

Pricing the Put Option

Similarly, the current value of the replicating portfolio for the put option is $V_{RP}^P = \Delta_P S + B_P$, where $\Delta_P = \dfrac{P_u - P_d}{S(u-d)} = \dfrac{-4.76}{50(.2003)} = -.475$ and $B_P = \dfrac{uP_d - dP_u}{e^{r\delta t}(u-d)} = \dfrac{1.1052(4.76)}{e^{.03}(.2003)} = \$25.47$. Therefore, we can replicate the put option by shorting .475 of a share for \$ 23.75 and lending \$ 25.47, which implies that $P = V_{RP} = -23.75 + 25.47 = \$1.72$.

3 Risk Neutral Valuation Approach in a Single Period

Pricing the Call and Put Options

Under risk neutral valuation, the pricing equation for a single time-step call option is $c = e^{-r\delta t}\left[qc_u + (1-q)c_d\right]$, and the pricing equation for a single time-step put option is $p = e^{-r\delta t}\left[qp_u + (1-q)p_d\right]$, where $q = \dfrac{e^{r\delta t} - d}{(u-d)}$. Thus, $q = \dfrac{e^{.03} - .9048}{(1.1052 - .9048)} = .627$, $c = e^{-.03}[.627(5.26)] = \$3.20$, and $p = e^{-.03}[.373(4.76)] = \$1.72$.

Problem 2: Now suppose that there are 2 timesteps, but the length of each timestep is 6 months rather than 1 year. What are the arbitrage-free values for a call and put based on these parameter values?

Under risk neutral valuation, the pricing equation for a two time-step call option is $c = e^{-r\delta t}\left[q^2 c_{uu} + 2q(1-q)c_{ud} + (1-q)^2 c_{dd}\right]$, and the pricing equation for a two time-step put option is $p = e^{-2r\delta t}\left[q^2 p_{uu} + 2q(1-q)p_{ud} + (1-q)^2 p_{dd}\right]$, where $q = \dfrac{e^{r\delta t} - d}{(u-d)}$, $u = e^{\sigma\sqrt{\delta t}} = e^{.1\sqrt{.5}} = 1.0733$, and $d = e^{-\sigma\sqrt{\delta t}} = e^{-.1\sqrt{.5}} = .9317$. Thus, $q = \dfrac{e^{.03/2} - .9317}{(1.0733 - .9317)} = .5891$, $c = e^{-.03}[.5891^2(7.50)] = \$2.56$, and $p = e^{-.03}[.4109^2(6.59)] = \$1.08$.