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Options, Futures and Other Derivatives — Name: SOLUTIONS
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Problem Set 9

Problem 1.
A company's cash position, measured in millions of dollars, follows a generalized Wiener process with a drift rate of 0.2 per month and a variance rate of 0.5 per month. The initial cash position is 3.0.

A. *What are the probability distributions of the cash position after one month, six months, and one year?*

B. *What are the probabilities of a negative cash position at the end of six months and one year?*

C. *At what time in the future is the probability of a negative cash position the greatest?*

A. The probability distributions are:

$$N(3.0 + 0.2,\ 0.5) = N(3.2, 0.5)$$

$$N(3.0 + 1.2,\ 0.5 \times 6) = N(4.2,\ 3)$$

$$N(3.0 + 2.4,\ 0.5 \times 12) = N(5.4,\ 6)$$

B. The chance of a random sample from $N(4.2,\ 3)$ being negative is

$$N\left(-\frac{4.2}{\sqrt{3}}\right) = N(-2.43),$$

where $N(x)$ is the cumulative probability that a standardized normal variable (i.e., a variable with probability distribution $N(0,\ 1)$) is less than x. From normal distribution tables $N(-2.43) = 0.0077$. Hence the probability of a negative cash position at the end of six months is 0.77%.

Similarly, the probability of a negative cash position at the end of one year is

$$N\left(-\frac{5.4}{\sqrt{6}}\right) = N(-2.20) = 0.0137 \text{ or } 1.37\%.$$

C. In general, the probability distribution of the cash position at the end of x months is

$$N(3.0 + 0.2x,\ 0.5x)$$

The probability of the cash position being negative is maximized when $\dfrac{3.0 + 0.2x}{\sqrt{0.5x}}$ is minimized. Define $y = \dfrac{3.0 + 0.2x}{\sqrt{0.5x}} = 4.243x^{-.5} + 0.283x^{.5}$. Then

$$\begin{aligned}\frac{dy}{dx} &= -2.1215x^{-1.5} + 0.1415x^{-.5} = 0 \\ \therefore 2.1215x^{-1.5} &= 0.1415x^{-.5} \\ x &= 2.1215/.1415 = 14.99.\end{aligned}$$

We know that this is a minimum since $d^2y/dx^2 = 3.18225x^{-2.5} + .07075x^{-1.5} > 0$ when $x = 14.99$.[1] Hence, the probability of a negative cash position is greatest after 15 months.

Problem 2.
An asset price is currently \$80. Its expected return and volatility are 8% and 25%, respectively.

A. What is the probability distribution for the annual continuously compounded rate of return earned on this asset?

SOLUTION: In this case, $\mu = 0.08$ and $\sigma = 0.25$. The probability distribution for the annual continuously compounded rate of return iassets:

$$\ln S_T/S \sim N(0.08 - \frac{0.25^2}{2},\ 0.25^2)$$

The expected value of the continuously compounded return is 4.875% per year and the standard deviation is 25% per year.

B. Suppose a European call option exists on this asset with an exercise price of \$80. The time to expiration on this option is two years. The riskless rate of interest is 3%. What is the value of this option and what is the risk neutral probability that this option will expire in the money?

SOLUTION: Applying the Black-Scholes-Merton option pricing formula, we find that the value of this call option is \$13.38. The risk neutral probability that this option will expire in the money corresponds to $N(d_2)$, where $d_2 = d_1 - \sigma\sqrt{T}$ and $d_1 = \dfrac{\ln(S/K) + (r + .5\sigma^2)T}{\sigma\sqrt{T}}$. Therefore, $d_2 = \dfrac{\ln(80/80) + (.03 + .5(.25)^2)2}{.25\sqrt{2}} - .25\sqrt{2} = -.0071 \Rightarrow$ $N(d_2) = 49.72\%$.

[1]Specifically, $d^2y/dx^2 = 0.00243883$ when $x = 14.99$.

C. What is the actual probability that this option will expire in the money?

SOLUTION: Over the next two years, the expected continuously compounded return is 9.75% and the standard deviation is $.25\sqrt{2} = 35.36\%$. The probability that the option will expire in the money is equal to

$$
\begin{aligned}
N\left(\frac{\ln(S/K) + (\mu - .5\sigma^2)\,T}{\sigma\sqrt{T}}\right) &= N\left(\frac{0 + (.08 - .5(0.03125))\,2}{.25\sqrt{2}}\right) \\
&= N\left(\frac{.0975}{.3536}\right) = .6086
\end{aligned}
$$

D. Why is there a difference between the two probabilities calculated in parts B and C above?

SOLUTION: The reason why the risk neutral probability is less than the actual probability is because the annualized and continuously compounded rate of return used under the risk neutral probability measure is $r = 3\%$, whereas the annualized and actual probability measure is $\mu - .5\sigma^2 = 4.875\%$. Since the actual probability measure is calculated using a higher rate of return, this translates into a higher probability compared with the risk neutral probability.[2]

[2]See my Actual versus Risk Neutral Probability of a Call Option Expiring in the Money teaching note for a detailed explanation as to *why* the risk neutral probability that a call option expires in-the-money must be less than the actual probability that this will occur.